This is part 2 (of 3) of the weekly homework. It is due Sunday August 13 at the review in SC A at 7 PM.

SUMMARY.

- $\text{curl}(P,Q,R) = (R_y - Q_z, P_z - R_x, Q_x - P_y)$ the **curl** of $F = (P,Q,R)$.
- The curl of a gradient field vanishes: $\text{curl}(\nabla f) = (0,0,0)$.
- $D : (u,v) \mapsto r(u,v) = (x(u,v), y(u,v), z(u,v))$ **surface** $S = r(D)$.
- $\int\int_S F \cdot dS = \int\int_R F(r(u,v)) \cdot (r_u \times r_v)\, dA$ **flux integral**.
- $\boxed{\int\int_S \text{curl}(F) \cdot dS = \int_C F \cdot dr}$ **Stokes theorem**, C: boundary of S, oriented so that "surface is to your left" if your head points in the normal direction.

Homework Problems

1) (4 points) Evaluate the flux integral $\int\int_S (0,0,yz) \cdot dS$, where S is the surface with parametric equation $x = uv, y = u+v, z = u-v$ on $R : u^2 + v^2 \leq 1$.

Solution:
$\vec{r}_u = (v,1,1), \vec{r}_v = (u,1,-1)$ so that $\vec{r}_u \times \vec{r}_v = (-2, u+v, -u+v)$. The flux integral is $\int\int_R (0,0,u^2-v^2) \cdot (-2, u+v, -u+v)\ dudv = \int\int_R v^2u - u^3 - v^3 + u^2v\ dudv$ which is best evaluated using polar coordinates: $\int_0^1 \int_0^{2\pi} r^4(\sin^2(\theta)\cos(\theta) - \cos^3(\theta) - \sin^3(\theta) + \cos^2(\theta)\sin(\theta))\ d\theta dr = 0$.

2) (4 points) Evaluate the flux integral $\int\int_S \text{curl}(F) \cdot dS$ for $F(x,y,z) = (xy, yz, zx)$, where S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $[1,0] \times [0,1]$ and has an upward orientation.

Solution:
$\text{curl}(F) = (-y,-z,-x)$. The parametrization $\vec{r}(u,v) = (u,v,4-u^2-v^2)$ gives $r_u \times r_v = (2u, 2v, 1)$ and $\text{curl}(F)(r(u,v)) = (-v, u^2+v^2-4, -u)$. The flux integral is $\int_0^1 \int_0^1 (-2uv + 2v(u^2+v^2-4) - u)\ dvdu = -1/2 + 1/3 + 1/2 - 4 - 1/2 = -25/6$.

3) (4 points) Evaluate the same flux integral as in the previous question but using Stokes theorem.

Hint. The boundary C can be built up using 4 curves:
$C_1 : \vec{r}(t) = (t, 0, 4-t^2), \vec{r}\,'(t) = (1,0,-2t)$.
$C_2 : \vec{r}(t) = (1, t, 3-t^2), \vec{r}\,'(t) = (0,1,-2t)$.
$C_3 : \vec{r}(t) = (t, 1, 4-t^2-1), \vec{r}\,'(t) = (1,0,-2t)$.
$C_4 : \vec{r}(t) = (0, t, 4-t^2), \vec{r}\,'(t) = (0,1,-2t)$.
Watch the orientation of each of these curves, when doing each line integral.

Solution:
The boundary C consists of the 4 curves given in the hint. The line integrals are:
$I : \int_0^1 (0,0,4t-t^3) \cdot (1,0,-2t)\ dt = -8/3 + 2/5 = -34/15$.
$II : \int_0^1 (t, 3t-t^3, 3-t^2) \cdot (0,1,-2t)\ dt = 1/4 - 3/2 = -5/4$.
$III : \int_0^1 (t, 3-t^2, 3t-t^3) \cdot (1,0,-2t)\ dt = -11/10$.
$IV : \int_0^1 (0, t(4-t^2), 0) \cdot (0,1,-2t)\ dt = 7/4$
$I + II - III - IV = -34/15 - 5/4 + 11/10 - 7/4 = -25/6$. (The line integrals III and IV along C_3, C_4 were taken negative because the curves are traced backwards.)

4) (4 points) Use Stokes theorem to evaluate $\int_C F \cdot dr$, where $F(x,y,z) = (x^2y, x^3/3, xy)$ and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$, oriented counterclockwise as viewed from above.

Solution:
1. Solution: The curl of F is $\text{curl}(F) = (x,-y,0)$. We can parametrize the hyperbolic paraboloid as $r(u,v) = (u\cos(v), u\sin(v), -u^2\cos(2v))$. $r_u \times r_v = (2u^2\cos(v), -2u^2\sin(v), u)$. $F(r(u,v)) = (u\cos(v), -u\sin(v), 0)$. $F(r(u,v)) \cdot (r_u \times r_v) = 2u^3$.

$$\int_0^1 \int_0^{2\pi} -2r^3\ d\theta dr = \pi\ .$$

2. Solution: with the parametrization $r(u,v) = (u, v, v^2 - u^2)$, we have $r_u \times r_v = (2u, -2v, 1)$ and $F(r(u,v)) = (u, -v, 0)$ so that $F(r(u,v)) \cdot (r_u \times r_v) = 2u^2 + 2v^2$. Integrating this over the disc gives $\boxed{\pi}$.

5) (4 points) If S is the surface $x^6 + y^6 + z^6 = 1$ and assume F is a smooth vector field. Show that $\int\int_S \text{curl}(F) \cdot dS = 0$.

Solution:
The flux of $\text{curl}(F)$ through a closed surface is zero by Stokes theorem and the fact that the surface does not have a boundary.
One can see this also by cutting the surface in two pieces and apply Stokes to both pieces.

Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

1) Solve Nashs problem distributed as an "in-class-exercice".

2) Use Stokes theorem to show that $\int_C (f\nabla g + g\nabla f) \cdot dr = 0$ for any closed curve C in space and any two functions f, g.
(Hint: the identity also follows from the fundamental theorem of line integrals).

3) Try to figure out, how Stokes theorem would look like in higher dimensions: in four dimensions, it is useful in special relativity.

Start: In dimension d, the curl is a field $\text{curl}(F)_{ij} = \partial_{x_j} F_i - \partial_{x_i} F_j$ with $\binom{d}{2}$ components. In 4 dimensions, it has 6 components. In d dimensions, a surface element in the $i-j$ plane is written as dS_{ij}. The flux integral of the curl of F through S is defined as $\int\int \text{curl}(F) \cdot dS$, where the dot product is $\sum_{i<j} \text{curl}(F)_{ij} dS_{ij}$. If S is given by a map r from a planar domain D to R^d, $U = \partial_u X$ and $V = \partial_v X$ are tangent vectors to that plane and $dS_{ij}(u,v) = (U_iV_j - U_jV_i)\ dudv$.