This is part 1 (of 2) of the weekly homework. It is due August 8 at the beginning of class.

SUMMARY.

- $\int\int\int_E f\ dV = \int_o^p \int_q^r \int_s^t f(x,y,z)\ dzdydx$ **triple integral**.
- $V(E) = \int\int\int_E 1\ dV$ **volume** of body E.
- $M(E) = \int\int\int_E \sigma\ dV$ **mass** of body E with density σ.
- $\frac{1}{V(E)} \int\int\int_E f\ dV$ **average value** of f over E.
- **Spherical coordinates** $\int\int\int_R g(\rho,\theta,\phi)\ \rho^2 \sin(\phi)\ d\rho d\theta d\phi$.
- **Cylindrical coordinates** $\int\int\int_E f(x,y,z)\ dxdydz = \int\int\int_R g(r,\theta,z)\ r\ dr\ d\theta\ dz$.

Homework Problems

1) (4 points) Evaluate the iterated integral $\int_0^1 \int_0^z \int_0^{2y} ze^{-y^2}\ dxdydz$.

Solution:
$\int_0^{2y} ze^{-y^2}\ dx = 2yze^{-y^2}$.
$\int_0^z 2yze^{-y^2}\ dy = z - ze^{-z^2}$.
$\int_0^1 z - ze^{-z^2}\ dz = z^2/2 - e^{-z^2}/2|_0^1 = \boxed{2e^{-1} - 1}$.

2) (4 points) Find the volume of the solid bounded by the paraboloids $z = x^2 + y^2$ and $z = 9 - (x^2 + y^2)$.

Solution:
Use cylindrical coordinates: The cylinders intersect when $r^2 = 9 - r^2$ or $r = \sqrt{9/2}$. We compute $\int_0^{\sqrt{9/2}} \int_0^{2\pi} (9 - r^2 - r^2) r\ d\theta dr = -2\pi \frac{(9-2r^2)}{2}|_0^{\sqrt{9}2} = \boxed{9\pi\sqrt{2}}$.

3) (4 points) Find the **moment of inertia** $\int\int\int_E (x^2 + y^2)\ dV$ of a cone $E = \{x^2 + y^2 \leq z^2\ 0 \leq z \leq 1\ \}$, when the cone is rotated around the z-axis.

Solution:
$2\pi \int_0^1 \int_0^z r^3\ drdz = 2\pi \int_0^1 z^4/4\ dz = \boxed{\pi/10}$.

4) (4 points) Sketch the solid whose volume is given by the integral $\int_0^{\pi/4} \int_0^{\pi/2} \int_0^3 \rho^2 \sin(\phi)\ d\rho d\theta d\phi$. Compute the integral.

Solution:
This is a quarter of an "ice cream cone".
The integral is $(\pi/2) \int_0^3 \rho^2\ d\rho \int_0^{\pi/2} \sin(\phi)\ d\phi = (\pi/2)9(1 - 1/\sqrt{2}) = \boxed{9\pi(2-\sqrt{2})/4}$.

5) (4 points) Find the mass of a solid hemi sphere of radius 1 if the density $\sigma(x,y,z)$ at any point (x,y,z) is equal to the distance from its base.

Remark. In other words, find $\int\int\int_E z\ dxdydz$, where E is the part of the unit sphere with $z \geq 0$.

Solution:
Use spherical coordinates:

$$\int_0^{2\pi} \int_0^{\pi/2} \int_0^1 (\rho\cos(\phi))\rho^2 \sin(\phi)\ d\rho d\phi d\theta$$

which evaluates to $2\pi \int_0^1 \rho^3\ d\rho \int_0^{\pi/2} \sin(2\phi)/2\ d\phi = 2\pi(1/4)(1/2) = \boxed{\pi/4}$.

Remarks

(You don't need to read these remarks to do the problems.)

The **moment of inertia** is defined as $I = \int\int\int_E r^2 \rho\ dV$ of a body E, where $r = r(x,y,z)$ is the distance of a point (x,y,z) to the axis of rotation and ρ is the density of the body.

The **moment of inertia** is the rotational analogue to **mass**.

Sphere	Hollow	$(2/3)MR^2$
Sphere	Solid	$(2/5)MR^2$
Cylinder	Solid (i.e. CD!)	$(1/2)MR^2$
Plate	Axes through center	$(1/12)M(a^2+b^2)$

If a body rotates with angular velocity ω, then $E = I\omega^2/2$ is the rotational energy of the body. The fact that fast rotation leads to a large energy can be demonstrated by spinning a CD until it disintegrates ... You find the movie on the website


Don't try that yourself!

Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

1) Find the volume of the intersection of two cylinders $y^2 + z^2 = 1$ and $x^2 + z^2 = 1$. Hint: Look what happens, when you cut the body at a fixed z value and calculate the area of this section.

2) Find a body which has as a xy-crosssection a circle of radius 1, as a xz-crosssection a rectangle of width 2 and height $\sqrt{3}$ and as a yz-crosssection an equilateral triangle of length 2. There are many bodies like that. Find one, for which you can compute the volume.

3) Use a quadruple integral to find the "hyper volume" of the hyper-sphere $x^2 + y^2 + z^2 + w^2 = r^2$ in $\mathbf{R}^4$.

Hint. If you slice up the hyper-sphere at w, you get a sphere of radius $\sqrt{r^2 - w^2}$. Integrate the volume of the sphere from $w = -r$ to $w = r$ using substitution.

4) Interpret the triple integral $\int\int\int_E f(x, y, z)\ dxdydz$ as the hyper volume of four dimensional region.