This is part 2 (of 2) of the weekly homework. It is due August 1 at the beginning of class.

SUMMARY. $dA = dxdy = rdrd\theta$ area element.

- $\int\int_R f(x,y)dxdy = \int_\alpha^\beta \int_a^b f(r\cos(\theta), r\sin(\theta))\, rdrd\theta$ **integral in polar coordinates**.
- $\int\int_R f(x,y)\, dxdy / \int\int_R 1\, dxdy$ is the **average value** of f on R.
- A curve $\vec{r}(t) = (f(t)\cos(t), f(t)\sin(t))$ can in polar coordinates (r,θ) be given as $r(\theta) = f(\theta)$.
- A vector valued function $\vec{r}(u,v)$ defines a **parametric surface** defined on a region R. It has the **surface area** $\int\int_R |\vec{r}_u(u,v) \times \vec{r}_v(u,v)|\, dudv$.

Homework Problems

1) (4 points) Integrate $f(x,y) = x^2$ over the unit disc $\{x^2 + y^2 \leq 1\}$ in two ways, first using Cartesian coordinates, then using polar coordinates.

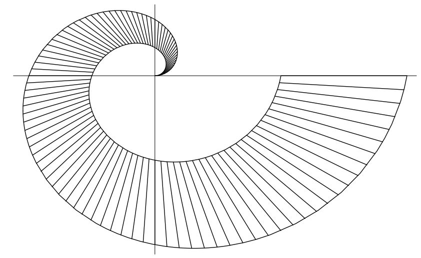
Solution:
The integral in Cartesian coordinates goes less smooth because we have to compute a 1D integral with partial integration: $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2\, dydx = \int_{-1}^1 2x^2\sqrt{1-x^2}\, dx = \pi/4$.
The integral in polar coordinates is easier to get: using the substitution $x = \cos(u)$ we obtain

$$\int_0^{2\pi} \int_0^1 r^2 \cos(\theta)^2 r\, drd\theta = (1/4)\pi\ .$$

2) (4 points) Find $\int\int_R (x^2+y^2)^{10}\, dA$, where R is the part of the unit disc $\{x^2+y^2 \leq 1\}$ for which $y > x$.

Solution:
The solution is easier in polar coordinates: $\int_0^1 \int_{\pi/4}^{5\pi/4} r^{21}\, d\theta dr = \pi/22$. This integral would give quite a bit of work to solve in Cartesian coordinates.

3) (4 points) What is the area of the region which is bounded by three curves, first by the polar curve $r(\theta) = \theta$ with $\theta \in [0, 2\pi]$, second by the polar curve $r(\theta) = 2\theta$ with $\theta \in [0, 2\pi]$ and third by the positive x-axis.



Solution:
$\int_0^{2\pi} \int_\theta^{2\theta} r\, drd\theta = \int_0^{2\pi} (2\theta)^2/2 - (\theta)^2/2\, d\theta = (3/2)\int_0^{2\pi} \theta^2\, d\theta = (3/2)(2\pi)^3/3 = 4\pi^3$.

4) (4 points) Find the average value of $f(x,y) = x^2 + y^2$ on the annulus $1 \leq |(x,y)| \leq 2$.

Solution:
The integral of f over the annulus is $2\pi \int_1^2 r^3\, dr = 2\pi(16-1)/4$. The area is $2\pi \int_1^2 r\, dr = 2\pi(4-1)/2$. The average is $(15/4)/(3/2) = 5/2$.

5) (4 points) Find the surface area of the paraboloid $x = y^2 + z^2$ that lies inside the cylinder $y^2 + z^2 = 9$.

Solution:
We use polar coordinates in the yz-plane. The paraboloid is parametrized by $(u,v) \mapsto (v^2, v\cos(u), v\sin(u))$ and the surface integral $\int_0^3 \int_0^{2\pi} |\vec{r}_u \times \vec{r}_v|\, dudv$ is equal to $\int_0^3 \int_0^{2\pi} v\sqrt{1+4v^2}\, dudv = 2\pi \int_0^3 v\sqrt{1+4v^2}\, dv = \pi(37^{3/2} - 1)/6$.

Challenge Problems

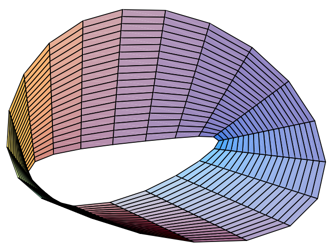
(Solutions to these problems are **not** turned in with the homework.)

1) The M"obius strip is a surface which has only one side. It is parametrized as $(1+(v-1/2)\cos(u/2))\cos(u), (1+(v-1/2)\sin(u/2))\sin(u), (v-1/2)\sin(u/2))$. What surface do you compute with the integral

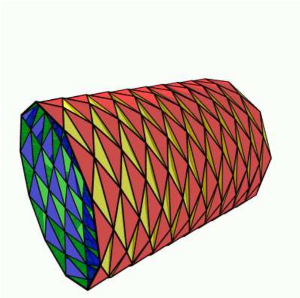
$$\int_0^{2\pi}\int_{-1}^{1} |\vec{r}_u(u,v)\times\vec{r}_v(u,v)|\, dudv\ ?$$

What surface do you compute with the integral

$$\int_0^{4\pi}\int_{-1}^{1} |\vec{r}_u(u,v)\times\vec{r}_v(u,v)|\, dudv\ ?$$



2) In class, you have seen a surface which encloses a finite volume and has infinite surface area. Can you construct for any constant M like $M = 10^{100} cm^2$ a surface inside the unit ball such that the surface area is bigger than M? The picture below should be a hint.



3) Calculate $\int\int_{\mathbf{R}^2} e^{-x^2-y^2}\, dxdy$ and use this to calculate the integral $\int_{-\infty}^{\infty} e^{-x^2}\, dx$.

Hint. The function $f(x) = e^{-x^2}$ is known to have no anti-derivative which can be expressed with "known functions" like $\exp, \log, \sin$ etc. You can nevertheless find a closed solution for the definite integral $\int_{-\infty}^{\infty} e^{-x^2}\, dx$.

4) Find the area of the region shaded in the picture. The region is bounded by the polar curves $r(\theta) = 2\theta$ with $\theta \in [0, 6\pi]$ and $r(\theta) = 3\theta$ with $\theta \in [0, 4\pi]$.