This is part 2 (of 3) of the weekly homework. It is due July 25 at the beginning of class.

SUMMARY.

- $\nabla f(x,y) = (0,0)$ **critical point** or **stationary point** (candidate for max or min).
- $H(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$ **Hessian**. $D = \det(H(x,y)) = f_{xx}f_{yy} - f_{xy}^2$ **determinant**, is called **discriminant**. (It is not necessary that you know determinants or matrices, you can survive with the formula $D = f_{xx}f_{yy} - f_{xy}^2$.
- $D > 0, f_{xx} < 0$ **local maximum**, $D > 0$, $f_{xx} > 0$ **local minimum**, $D < 0$ **saddle point**.

Homework Problems

1) (4 points) Find all the extrema of the function $f(x,y) = x^3 - 3x + 2y^2 - y^4$ and determine whether they are maxima, minima or saddle points.

Solution:
The condition $\nabla f(x,y) = (f_x, f_y) = (0,0)$ are the equations

$$3x^2 - 3 = 0 \quad (1)$$
$$4y - 4y^3 = 0 \quad (2)$$

have the solution $x = \pm 1$, $y = 0$ or $y = \pm 1$.

Point	D	f_{xx}	nature
(-1,-1)	48	-6	local maximum
(-1,0)	-24	-6	saddle
(-1,1)	48	-6	local maximum
(1,-1)	-48	6	saddle point
(1,0)	24	6	local minimum
(1,1)	-48	6	saddle point

2) (4 points) Find all the extrema of the function $f(x,y) = x^2 - y^2 + xy$ and determine whether they are maxima, minima or saddle points.

Solution:
The condition $\nabla f(x,y) = (f_x, f_y) = (0,0)$ are the equations

$$2x + y = 0 \quad (3)$$
$$-2y + x = 0 \quad (4)$$

which have only the solution $(0,0)$. Because $D = -4 - 1^2 < 0$ this is a saddle point.

3) (4 points) Find and classify all the extrema of the function $f(x,y) = e^{-x^2-y^2}(x^2 + 2y^2)$.

Solution:
We have to solve the equations $\nabla f = (2^2+2y^2-1)xe^{-x^2-y^2}, (x^2+2y^2-2)ye^{-x^2-y^2}) = (0,0)$ which has the solutions $x = 0, y = 0$, $x = 0, y = \pm 1$, $y = 0, x = \pm 1$. By evaluating D and f_{xx} at each point we find their nature:

Point	D	f_{xx}	nature
(0,0)	8	2	local minimum
(0,1)	$16/e^2$	-2/e	local maximum
(0,-1)	$16/e^2$	-2/e	local maximum
(1,0)	$-8/e^2$	-4/e	saddle point
(-1,0)	$-8/e^2$	-4/e	saddle point

4) (4 points) Find the critical point of the function $f(x,y) = x^2 + ky^2 + xy$ for general k. Describe what happens with that critical point and its nature (maximum, minimum or saddle) as the parameter k changes. Especially, at which values of k does the nature of this critical point change?

Solution:
The equations

$$2x + y = 0$$
$$2yk + x = 0$$

can be solved by substituting $y = -2x$ into the second equation: $-4xk + x = 0$ has for all k the solution $x = 0$ leading to $y = 0$. So, $(0,0)$ is a critical point for all k.
At that critical point the discriminant is $D = f_{xx}f_{yy} - f_{xy}^2 = 4k - 1$. For $k < 1/4$, we have $D < 0$ and the critical point is a saddle. For $k > 1/4$, we have $D > 0$ and $f_{xx} > 0$ so that the critical point is a local minimum. At the parameter value $k = 1/4$, the nature of the critical point changed.

5) (4 points)
Consider the function $f(x,y) = x^4 - y^4$. The graph of f looks like a saddle, but it is quite "flat". The point $(0,0)$ is a critical point as you can verify by computing the gradient at $(0,0)$. In this problem, you see that this critical point can be perturbed in any way. You can change the function a bit to achieve a local maximum, to get a local minimum or to get a saddle point. The nature of the critical point is not "robust" in this case.

a) Add a function $g(x,y)$ so that $f(x,y) + \epsilon g(x,y)$ has a local maximum at $(0,0)$ for all $\epsilon > 0$.

b) Add a function $g(x,y)$ so that $f(x,y) + \epsilon g(x,y)$ has a local minimum at $(0,0)$ for all $\epsilon > 0$.

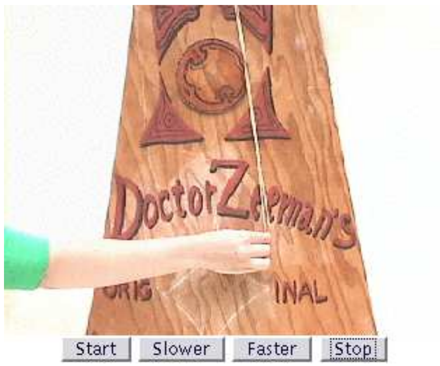
c) Add a function $g(x,y)$ so that $f(x,y) + \epsilon g(x,y)$ has a saddle point at $(0,0)$ for all $\epsilon > 0$.

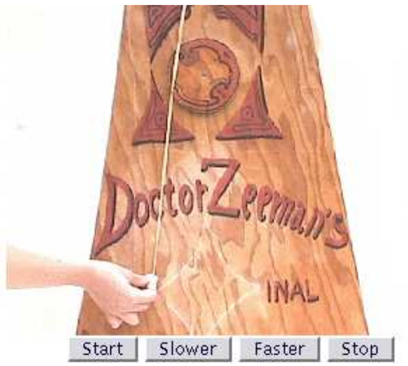
Solution:
a) Take for example $g(x,y) = -(x^2 + y^2)$.
b) Take for example $g(x,y) = (x^2 + y^2)$
c) Take for example $g(x,y) = x^2 - y^2$.
For whatever small $\epsilon > 0$, if we look at $x^4 - y^4 + \epsilon g(x,y)$, the $g(x,y)$ term dominates near $(0,0)$.

Remarks

(You don't need to read these remarks to do the problems.)

To Problem 4). The constant k is here called a **parameter**. Parameter values of k, where the nature of the critical point changes, are called **bifurcation parameters**. They play an important role in physics. If more then one parameters are involved there is a whole theory called **catastrophe theory** which deals with bifurcations. Bifurcations are points, where qualitative changes in the system happen. These can often be catastrophes. For example, for two parameters the the most common type of catastrophes is called a "cusp" catastrophe which can be illustrated with a **catastrophe machine**. (Try it out at http://www.ams.org/new-in-math/cover/cusp4.html).





Catastrophe theory has been applied to a number of different phenomena, such as the stability of ships at sea and their capsizing, bridge collapse, fight-or-flight behavior of animals or even prison riots.

Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

1) Find and classify all critical points of $\log(200+\log(100+\log(200+3x^2+xy+2x+y^2+y+4)))$.

2) How would a classification of critical points in three dimensions look like. Formulate a criterion for local maxima or local minima for functions $f(x,y,z)$ of three variables.

3) Can you prove the "Island theorem" on the handout?