7/6/2006 IMPLICIT AND PARAMETRIC SURFACES   O.Knill,Maths21a

This is part 2 (of 2) of the homework which is due July 11 at the beginning of class.

SUMMARY:

- $g(x,y,z) = 0$ **implicit surface**.

- $x = r\cos(\theta)$
  $y = r\sin(\theta)$
  $z = z$
  cylindrical coordinates

- $x = \rho\sin(\phi)\cos(\theta)$
  $y = \rho\sin(\phi)\sin(\theta)$
  $z = \rho\cos(\phi)$
  spherical coordinates

- $\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$
  defines a **parametric surface**.

- Switching from implicit to parametric is not trivial in general. For spheres, planes, surfaces of revolution and graphs we can do it.

EXAMPLES:

- $x^2 + y^2 + z^2 = \rho^2$ sphere

- $r = 1$, **cylinder**.

- $\rho = 1$, **sphere**.

- $r = z$, **cone**

- $\vec{r}(u,v) = P + u\vec{u} + v\vec{v}$ **plane**

- $\vec{r}(u,v) = (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$ **sphere**

- $\vec{r}(u,v) = (\cos(u), \sin(u), v)$ **cylinder**

- $\vec{r}(u,v) = (u, v, f(u,v))$ **graph** of f

Homework Problems

1) (4 points)
a) What is the equation for the surface $x^2 + y^2 - 5x = z^2$ in cylindrical coordinates?
b) Describe in words or draw a sketch of the surface whose equation is $\rho = \sin(\phi)$ in spherical coordinates (ρ, θ, ϕ).

Solution:
a) $r^2 - 5r\cos(\theta) = z^2$.
b) This is a fat bagel.

2) (4 points) Plot the surface with the parametrization $\vec{r}(u,v) = (v^2\cos(u), v^2\sin(u), v)$, where $u \in [0, 2\pi]$ and $v \in \mathbf{R}$.

Solution:
It is a surface of revolution, very thin at the origin. The shape is a parabola but it is bent the other way round as in the paraboloid.

3) (4 points) Find a parametrization for the plane which contains the points $(3,4,1),(1,2,1)$ and $(0,3,4)$.

Solution:
$\vec{r}(s,t) = (3 - 2s - 3t, 4 - 2s - t, 1 + 3t)$.

4) (4 points) Find two different parametrisations of the lower half of the ellipsoid $2x^2 + 4y^2 + z^2 = 1$.
Hint: for one of the parametrizations assume that the surface is a graph. For the other, do something similar than for the sphere.
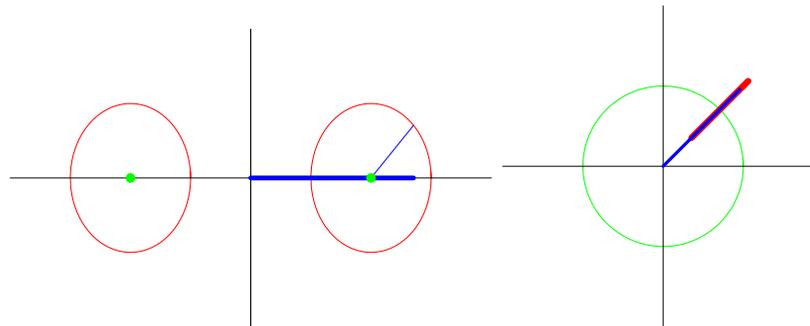
Solution:
Here are three possible parametrizations:
1) $\vec{r}(u,v) = (u, v, -\sqrt{1 - 2u^2 - 4v^2})$.
2) $\vec{r}(\theta, \phi) = (\sin(\phi)\cos(\theta)/\sqrt{2}, \sin(\phi)\sin(\theta)/2, \cos(\phi))$.

5) (4 points) Find a parametrisation of the **torus** which is obtained as the set of points which have distance 1 from the circle $(2\cos(\theta), 2\sin(\theta), 0)$, where θ is the angle occuring in cylindrical and spherical coordinates.

Hint: Keep $u = t$ as one of the parameters and let r the distance of a point on the torus to the z-axis. This distance is $r = 2 + \cos(\phi)$ if ϕ is the angle you see on Figure 1. You can read off from the same picture also $z = \sin(\phi)$. To finish the parametrization problem, you have to translate back from cylindrical coordinates $(r, \theta, z) = (2 + \cos(\phi), \theta, \sin(\phi))$ to Cartesian coordinates (x, y, z). Write down your result in the form $\vec{r}(\theta, \phi) = (x(\theta, \phi), y(\theta, \phi), z(\theta, \phi))$.

Solution:
$\vec{r}(\theta, \phi) = ((2 + \cos(\phi))\cos(\theta), (2 + \cos(\phi))\sin(\theta), \sin(\phi))$.

Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

1) Try to graph without a computer the surface $r = f(\theta, \phi) = (2 + \sin(3\theta))(2 + \cos(2\phi))$ (It is a **graph** in spherical coordinates (r, ϕ, θ).)

Hint. Do it in stages. First graph $r = 2$ (the sphere), then $r = (2 + \sin(3\theta))$, then draw a sketch of the final surface.

2) How would you design analogues of spherical or cylindrical coordinates in 4 dimensions?

3) Sketch the surface $r(u,v) = (2 + 2v\cos(\pi u))\cos(2\pi u), (2 + 2v\cos(\pi u))\sin(2\pi u), v\sin(\pi u))$.

4) The **torus** is obtained by bending and gluing the ends of a cylinder together.
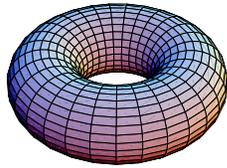
The **Klein bottle** is obtained in the same way, however, the ends are put together with opposite directions. This can not be achieved without self-intersection. Take one end of the tube, bend it, enter the tube first to match the ends in the opposite direction as for the torus. Can you find a parametrisation $r(u, v)$ for this surface? On the handout for this lecture, you find a parametrization of the same surface but which looks different. The idea is to have a parametrization which produces the "bottle".