This is part 1 (of 2) of the homework which is due July 11 at the beginning of class.

SUMMARY.

- $f(x, y), g(x, y, z)$ **functions of several variables**.
- $\{f(x, y) = c\}$ are **curves** called **contour curves**.
- $\{g(x, y, z) = d\}$ are **surface** called **contour surface**.
- $\{(x, y, z) = f(x, y))\}$ **graph** of function $f(x, y)$. It is a surface.
- **trace**: intersection of graph with coordinate planes.
- **intercepts**: intersection of graph with coordinate axis.

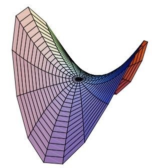
**Quadrics:**

- $x^2 + y^2 + z^2 = 1$ **sphere**.
- $x^2 + y^2 - z^2 = 1$ **one-sheeted hyperboloid**.
- $x^2 + y^2 - z^2 = -1$ **two-sheeted hyperboloid**.
- $x^2 + y^2 - z^2 = 0$ **cone**.
- $x^2 + y^2 = z$ **paraboloid**.
- $x^2 - y^2 = z$ **hyperbolic paraboloid**.
- $x^2 + y^2 = 1$ **cylinder**.

## Homework Problems

1) (4 points) Let $f(x, y) = x^4 - y^4$.
a) (1) Write the graph of $f$ as a contour surface $g(x, y, z) = 0$.
b) (1) Find the equations for the three traces of $g$.
c) (2) Sketch the graph of $f$.

**Solution:**
a) $z = f(x, y)$ gives $g(x, y, z) = z - f(x, y) = z - x^4 + y^4 = 0$.
b) The $xy$-trace: $x^4 - y^4 == (x^2 - y^2)(x^2 + y^2)$ is a union of two lines.
The $yz$-trace: $z + y^4 = 0$ is a "fat parabola".
Also the $xz$-trace: $z - x^2 = 0$ is a "fat parabola".
c) The surface has the same shape as the hyperbolic paraboloid, a "pringle".



2) (4 points) Consider the surface $z^2 + x^2 - 2x - y = 0$. Draw the three traces. What surface is it?

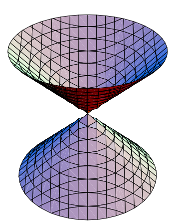
**Solution:**
Completion of the sphere gives $z^2+(x-1)^2-y = 1$. The surface is a paraboloid rotational symmetric parallel to the $y$ axis. To see this, it is helpful to draw the generalized traces obtained by intersecting with $y = c$ which gives circles. The other two traces are parabola.

3) (4 points) Surfaces satisfying the implicit equation $x^k + y^k = z^k$ with integer $k$ are called **Fermat** surfaces. In this exercice, we don't want you to compute, but to draw and visualize. Traces help you to get the picture right.

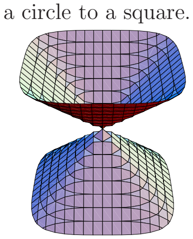
a) Sketch the Fermat surface for $k = 2$ with traces.
b) Sketch the Fermat surface for $k = 4$ with traces.

**Solution:**

The surface $x^2 + y^2 = z^2$ is a cone.



The surface $x^4+y^4 = z^4$ has on each height $z$ a trace which has the shape when you deform a circle to a square.



4) (4 points)

a) (1) Sketch the graph of the function $f(x, y) = 1/(x^2 + 4y^2)$.
b (1) Sketch the graph of the function $g(x, y) = |x| + |y|$.
c) (1) Sketch some contour curves $f(x, y) = c$ of $f$.
d) (1) Sketch some contour curves $g(x, y) = c$ of $g$.

**Solution:**
a) A singular spike of elliptic shape.
b) In each quadrant, the graph is linear and a plane.
These four pieces come together above the cordinate axis.
c) These are ellipses.
d) These are romboids, by 45 degree turned squares.

5) (4 points) Verify that the line $\vec{r}(t) = (1, 3, 2) + t(1, 2, 1)$ is contained in the surface $y = z^2 - x^2$.

**Solution:**
Plug in $x = 1 + t, y = 3 + 2t, z = 2 + t$ into the equation $(3 + 2t) = (2 + t)^2 - (1 + t)^2$.
The surface is actually a hyperbolic paraboloid.

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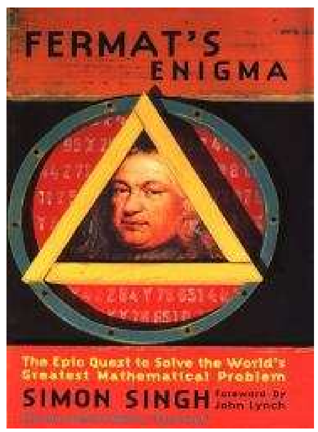
## Remarks

(You don't need to read these remarks to do the problems.)

To problem 3: You have found integer points $(x, y, z)$ lying on the Fermat surface $x^2 + y^2 = z^2$ in a previous homework. It was Fermat, who conjectured first, that there are no nontrivial lattice points on the Fermat surfaces for $n > 2$. This claim is now a theorem. It has been proven only a few years ago. The story about Fermat Last Theorem is told in many places. A nice little book is "The Fermat Enigma" by Simon Singh.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## Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

1) Find as many curves as you can which are obtained by intersecting two quadrics. To get full credit for this problem you have to provide at least three different curves.

2) How would you visualize the graph of a function $f(x, y, z)$ of three variables? How would you describe the set $\{(x, y, z, u) \mid g(x, y, z, u) = d\}$, where $g$ is a function of 4 variables and $d$ is a constant? Take the example of the four-dimensional sphere $x^2 + y^2 + z^2 + u^2 = 1$.