This is part 3 (of 3) of the homework which is due Wednesday July 5 at the beginning of class.

SUMMARY.

- $\vec{v} \times \vec{w} = \langle v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1 \rangle = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$. **cross product**.
- $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}|\sin(\alpha)$, where α is the **angle** between vectors. **Area** of parallelogram spanned by $\vec{v}$ and $\vec{w}$.
- $\vec{v} \times \vec{w}$ is **orthogonal** to $\vec{v}$ and to $\vec{w}$ with length $|\vec{v}||\vec{w}|\sin(\phi)$
- $\vec{u} \cdot (\vec{v} \times \vec{w})$ **triple scalar product**, signed volume of parallelepiped spanned by $\vec{u}, \vec{v}, \vec{w}$.
- $\vec{r}(t) = P + t\vec{v}$ **parametric equation** for a line, P a point, $\vec{v}$ is a vector.
- $\frac{(x-x_0)}{a} = \frac{(y-y_0)}{b} = \frac{(z-z_0)}{c}$ **symmetric equation** for a line.
- $\vec{r}(t,s) = P + t\vec{v} + s\vec{w}$ **parametric equation** for a **plane**.
- $ax + by + cz = d$, **implicit equation** for a **plane**.
- Distance Point-Point $d(P,Q) = |\vec{PQ}|$.
- Distance Point-Plane $d(P,\Sigma) = |(\vec{PQ}) \cdot \vec{n}|/|\vec{n}|$.
- Distance Point-Line $d(P,L) = |(\vec{PQ}) \times \vec{u}|/|\vec{u}|$.
- Distance Line-Line $d(L,M) = |(\vec{PQ}) \cdot (\vec{u} \times \vec{v})|/|\vec{u} \times \vec{v}|$.

Homework Problems

1) (4 points)
a) (2) Find a the cross product $\vec{w}$ of $\vec{u} = (-3,-1,2)$ and $\vec{v} = (-2,-2,3)$.
b) (1) Find a unit vector $\vec{n}$ orthogonal to $\vec{u}$ and $\vec{v}$.
c) (1) Find the volume of the parallelepiped spanned by $\vec{u}, \vec{v}$ and $\vec{w}$.

Solution:
a) $\vec{u} \times \vec{v} = \vec{w} = (1,5,4)$
b) $|\vec{w}| = 42$, $\vec{n} = \vec{w}/|\vec{w}| = (1/\sqrt{42}, 5/\sqrt{42}, 4/\sqrt{42})$.
c) $\vec{w} \cdot (\vec{u} \times \vec{v}) = \vec{w} \cdot \vec{w} = |\vec{w}|^2 = 42$.

2) (4 points)
a) (2) Find the distance between the point $(2,-1,2)$ and the plane $4x - 2y + z = 2$.
b) (2) A regular tetrahedron has vertices at the points $P_1 = (0,0,3)$,$P_2 = (0,\sqrt{8},-1)$, $P_3 = (-\sqrt{6},-\sqrt{2},-1)$ and $P_4 = (\sqrt{6},-\sqrt{2},-1)$. Find the distance between two edges which do not intersect.

Solution:
a) The point $Q = (0,0,2)$ is on the plane. The scalar projection of $P - Q = (2,-1,0)$ onto the normal vector $(4,-2,1)$ of the plane is $10/\sqrt{21}$.
b)The vector $\vec{v} = (P_2 - P_1) = (0, 2\sqrt{2}, -4)$ is parallel to the first edge, the vector $\vec{w} = (P_4 - P_3) = (2\sqrt{6}, 0, 0)$ is parallel to the second edge. The cross product of $\vec{v}$ and $\vec{w}$ is $\vec{n} = (0, -8\sqrt{6}, -8\sqrt{3})$. The distance between the two edges is the scalar projection of $P_3 - P_1$ onto $\vec{n}$. It is $(P_3 - P_1).\vec{n}/|\vec{n}| = 2\sqrt{3}$.

3) (4 points)
a) Assume $\vec{u} + \vec{v} + \vec{w} = \vec{0}$. Verify that $\vec{u} \times \vec{v} = \vec{v} \times \vec{w} = \vec{w} \times \vec{u}$.
b) Find $(\vec{u} + \vec{v}) \cdot (\vec{v} \times \vec{w})$ if $\vec{u}, \vec{v}, \vec{w}$ are unit vectors which are orthogonal to each other and $\vec{u} \times \vec{v} = \vec{w}$.
c) Assume you have a triangle in the plane which has edge points with integer coordinates. Show that the area of the triangle is an integer, or half of an integer.

Solution:
a) Build a triangle with the three vectors u, v, w. Each of the terms in the identity is twice the area of the triangle.
b) The result is 1.
c) The formula for the area is $|(A - C) \times (B - C)|/2$.

4) (4 points) Given three vectors $\vec{u}, \vec{v}, \vec{w}$ with $V = \vec{u} \cdot (\vec{v} \times \vec{w}) \neq 0$. Define three new vectors

$$\begin{aligned} \vec{a} &= (\vec{v} \times \vec{w})/V \\ \vec{b} &= (\vec{w} \times \vec{u})/V \\ \vec{c} &= (\vec{u} \times \vec{v})/V \ . \end{aligned}$$

Verify that $\vec{a} \cdot (\vec{b} \times \vec{c}) = 1/V$.

Hint. Use the identity $\vec{b} \times (\vec{u} \times \vec{v}) = (\vec{b} \cdot \vec{v})\vec{u} - (\vec{b} \cdot \vec{u})\vec{v}$. which you don't need to derive.

Solution:
Focus first on $(\vec{b} \times \vec{c}) = \vec{b} \times (\vec{u} \times \vec{v})/V$ and use the hint to get $(\vec{b} \cdot \vec{v})\vec{u}/V - (\vec{b} \cdot \vec{u})\vec{v}/V$. The problem asks for the dot product of this with $\vec{a}$. Now, since $\vec{a}$ is orthogonal to the second term, we obtain $\vec{a}(\vec{b} \cdot \vec{v}) \cdot \vec{u}/V$. When plugging in the definitions of $\vec{a}$ and $\vec{b}$, we are left with $V^2/V^3 = 1/V$.

5) (4 points)
a) (2) Find the parametric and symmetric equation for the line which passes through the points $P = (1,2,3)$ and $Q = (3,4,5)$.
b) (2) Find the equation for the plane which contains the three points $P = (1,2,3)$, $Q = (3,4,4)$ and $R = (1,1,2)$.

Solution:
a) The vector $\vec{v} = (2,2,2)$ connects the two points. The parametric equation is $P + t\vec{v} = (1,2,3) + t(2,2,2) = (1+2t, 2+2t, 3+2t)$. The symmetric equation is $(x-1)/2 = (y-2)/2 = (z-3)/2$.
b) A normal vector $\vec{n} = (1,-2,2) = (a,b,c)$ of the plane $ax + by + cz = d$ is obtained as the cross product of $P - Q$ and $R - Q$ With $d = \vec{n} \cdot P = 3$, we have the equation $x - 2y + 2z = 3$.
b0

Remarks

(You don't need to read these remarks to do the problems.)

To problem 4): three vectors whose triple scalar product is not zero are called **non-coplanar** . Adding integer multiples of such vectors form a **lattice**. The points of the lattice are all points $n\vec{u} + m\vec{v} + k\vec{w}$, where n, m, k are integers.
The three new vectors $\vec{a}, \vec{b}$ and $\vec{c}$ defined in problem 5) define a new lattice which is called the **reciprocal lattice**. Crystallographers also denote them by $\vec{u}^*, \vec{v}^*$ and $\vec{w}^*$. What you have shown in 5) is that the volume V^* of the unit cells of the reciprocal lattice is the inverse $1/V$ of the volume V of the unit cell of the lattice itself. The reciprocal lattice is essential for the study of crystal lattices and their diffraction properties which can be measured by shooting X-rays onto them. A convenient way to link the structure of the material to it's diffraction pattern is through the reciprocal lattice.

Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

1) Prove the following identity for vectors $\vec{a}, \vec{b}, \vec{c}$ in space:

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

2) Find a general formula for the volume of a tetrahedron with corners P, Q, R, S.
Hint. Find first a formula for the area of one of its triangular faces, and then a formula for the distance from the fourth point to that face.

3) The change of the angular momentum satisfes the formula

$$\frac{d}{dt}\vec{L} = \vec{L} \times \vec{\Omega} ,$$

where $\vec{\Omega}$ is the angular velocity vector. Verify that the length of $\vec{L}$ does not change in time.