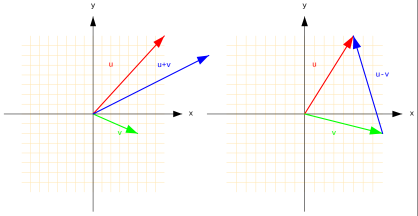
This is part 2 (of 3) of the homework. It is due Wednessday July 5 at the beginning of class. Note that all the homework assigned the first week is due on Wednesday!

SUMMARY.
Vectors $\vec{v} = \langle v_1, v_2, v_3 \rangle = \vec{PQ}$.
Points $\vec{v} = \vec{OP} = \vec{P}$, with $O = (0,0,0)$.
$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$ **dot product**
$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}|\cos(\alpha)$, where α is the **angle**.
$\vec{\mathrm{proj}}_{\vec{w}}(\vec{v}) = \vec{w}(\vec{v} \cdot \vec{w})/|\vec{w}|^2$ **projection** of $\vec{v}$ onto $\vec{w}$.
$\mathrm{comp}_{\vec{w}}(\vec{v}) = |\vec{\mathrm{proj}}_{\vec{w}}(\vec{v})| = (\vec{v} \cdot \vec{w})/|\vec{w}|$
component



Homework Problems

1) (4 points) Let $\vec{u} = (2,3)$ and $\vec{v} = (-2,1)$.
a) Draw the vectors $\vec{u}, \vec{v}, \vec{u}+\vec{v}, \vec{u}-\vec{v}$.
b) What is the relation between the length of $\vec{u}-\vec{v}$ and the lengths of $\vec{u}$ and $\vec{v}$?
c) Prove the Pythagoras theorem: if $\vec{u}, \vec{v}$ are orthogonal $\vec{u} \cdot \vec{v} = 0$, then $|\vec{u}-\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2$.

Solution:
a) draw a parallelogram. $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ are diagonals.
b) $|\vec{u}-\vec{v}| \leq |\vec{u}| - |\vec{v}|$.
c) $(\vec{u}-\vec{v}) \cdot (\vec{u}-\vec{v}) = \vec{u} \cdot \vec{u} - 2\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v} = |\vec{u}|^2 + |\vec{v}|^2$, where we used $\vec{u} \cdot \vec{v} = 0$.

2) (4 points)
a) Verify that if s, t are two integers, then $\vec{v} = \langle x, y \rangle$ is a vector with integer length if $x = 2st, y = t^2 - s^2$. This gives **Pythagorean triples** $x^2 + y^2 = z^2$.
b) An **Euler brick** is a cuboid of dimensions a,b,c such that the face diagonals are integers. Verify that $\langle a, b, c \rangle = \langle u(4v^2 - w^2), v(4u^2 - w^2), 4uvw \rangle$ leads to an Euler brick if $u^2 + v^2 = w^2$.
c) Verify that $\vec{v} = \langle a, b, c \rangle = \langle 240, 117, 44 \rangle$ is a vector which leads to an Euler brick.

P.S. If also the space diagonal $\sqrt{a^2 + b^2 + c^2}$ is an integer, an Euler brick is called a **perfect cuboid**. It is an open mathematical problem, whether a perfect cuboid exists. Nobody has found one, nor proven that it can not exist.

Solution:
a) $(x, y) = (2st, t^2 - s^2)$ has length $\sqrt{4s^2t^2 + t^4 + s^4 - 2s^2t^2} = (t^2 + s^2)$.
b) We have to verify that $a^2 + b^2$ and $b^2 + c^2$ and $a^2 + c^2$ are all squares

$$\begin{aligned} a^2 + b^2 &= u^2(4v^2 - w^2)^2 + v^2(4u^2 - w^2)^2 = u^6 + 3u^4v^2 + 3u^2v^4 + v^6 = (u^2 + v^2)^3 = w^6 \\ b^2 + c^2 &= v^2(4u^2 - w^2)^2 + (4uvw)^2 = v^2(16u^4 - 8u^2w^2 + w^4 - 16u^2w^2) = v^2(4u^2 + w^2)^2 \\ a^2 + c^2 &= u^2(4v^2 - w^2)^2 + (4uvw)^2 = u^2(16v^4 - 8v^2w^2 + w^4 - 16v^2w^2) = u^2(4v^2 + w^2)^2 \end{aligned}$$

c) $240^2 + 117^2 = 267^2$, $117^2 + 44^2 = 125^2$, $240^2 + 117^2 = 244^2$.

3) (4 points) **Colors** are encoded by vectors $\vec{v} = (r, g, b)$, where the **red**, **green** and **blue** components are all numbers in the interval $[0, 1]$. Examples are:

(0,0,0)	black	(0,0,1)	blue
(1,1,1)	white	(1,1,0)	yellow
(1/2,1/2,1/2)	gray	(1,0,1)	magenta
(1,0,0)	red	(0,1,1)	cyan
(0,1,0)	green	(1,1/2,0)	orange
(0,1,1/2)	spring green	(1,1,1/2)	khaki
(1,1/2,1/2)	pink	(1/2,1/4,0)	brown

a) Determine the angle between the colors magenta and cyan.
b) Find a color which is both orthogonal to orange and yellow.
c) What does the scaling $\vec{v} \mapsto \vec{v}/2$ do, if $\vec{v}$ represents a color?
d) If $\vec{v}$ and $\vec{w}$ are two colors, their mixture $(\vec{v} + \vec{w})/2$ is also a color. What is the mixture of red and white?
e) Vectors on the diagonal $r = g = b$ are called **gray** colors. Find the gray vector which is the vector projection of yellow onto white.

Solution:
a) $\cos(\alpha) = \frac{(1,0,1)\cdot(0,1,1)}{(|(1,0,1)||(0,1,1)|)} = 1/2$ gives $\alpha = 60^\circ = \pi/3$.
b) Blue $\vec{b} = (0,0,1)$ is orthogonal to orange $\vec{y} = (1, 1/2, 0)$ and yellow.
c) It darkens the color.
d) Pink.
e) The vector projection of yellow $\vec{y} = \langle 1,1,0 \rangle$ onto white $\vec{w} = \langle 1,1,1 \rangle$ is $\vec{w}(\vec{y} \cdot \vec{w})/|\vec{w}|^2 = \langle 2/3, 2/3, 2/3 \rangle$.

4) (4 points)
a) Find the angle between the diagonal of the unit cube and one of the diagonal of one of its faces. Assume that the two diagonals go through the same edge of the cube.
Remark. You can leave the answer in the form $\cos(\alpha) = ...$
b) Find the angle between two face diagonals which go through the same edge and are an adjacent faces.

Solution:
a) Put the coordinate system so that

$$(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (0,1,1), (1,1,1)$$

are the corners of the cube. The main diagonal is the vector $\vec{v} = (1,1,1)$, a diagonal in the face is $\langle 1,1,0 \rangle$. Then, $\cos(\alpha) = 2/(\sqrt{3}\sqrt{2})$.
b) We can take the two diagonals $(1,1,0), (0,1,1)$ and the angle satisfies $\cos(\alpha) = 1/2$ so that $\alpha = \pi/3$.

5) (4 points) Assume $\vec{v} = \langle -4, 2, 2 \rangle$ and $\vec{w} = \langle 3, 0, 4 \rangle$.

a) Find the length of $\vec{w}$ and the dot product between $\vec{v}$ and $\vec{w}$.
b) Find the vector projection of $\vec{v}$ onto $\vec{w}$.
c) Find the component of $\vec{v}$ on $\vec{w}$.
d) Find a vector parallel to $\vec{w}$ of length 1.

Solution:
a) $|\vec{w}| = 5$, $\vec{v} \cdot \vec{w} = -4$.
b) $\vec{w}(\vec{v} \cdot \vec{w})/|\vec{w}|^2 = -4/25(3,0,4) = (-12/25, 0, -16/25)$.
c) $(\vec{v} \cdot \vec{w})/|\vec{w}| = -4/5$.
d) $\vec{w}/|\vec{w}| = (3/5, 0, 4/5)$.

Remarks

(You don't need to read these remarks to do the problems.)

To problem 2) The construction of Pythagorean triples has been known already by the Babylonians. The construction is vital, because it allowed (using a rope only) to construct precise right angles or measure area. It is an early example, where a mathematical discovery lead to the enhancement of economics. Measuring area for example was crucial for the trading of land.

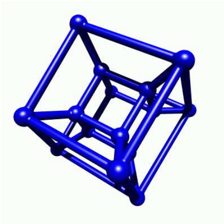
To problem 3) In many computer applications, color is encoded in hexadecimal form, where r,g,b are integers from 1 to 255. For example, the word fa887b indicates the color, where the red component is $fa = 11 + 15 \cdot 16 = 251$, green is $88 = 8 \cdot 16 + 8 = 136$, blue $= 7b = 7 \cdot 16 + 12 = 124$. This color corresponds to the vector $(251/255, 136/255, 124/255)$ in the unit cube. You see expressions like "$bgcolor = \#fa887b$" in HTML source code of web pages.

Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

1) Verify that for any two vectors $\vec{a}$ and $\vec{b}$, the inequality $|\vec{a} - \vec{b}| \geq ||\vec{a}| - |\vec{b}||$ holds.

2) The coordinates for the corners of a cube in 4D are the 16 points $(\pm 1, \pm 1, \pm 1, \pm 1)$. Find the angle between the big diagonal connecting $(1,1,1,1)$ with $(-1,-1,-1,-1)$ and the "middle diagonal" in one of 3D faces connecting $(1,1,1,1)$ with $(-1,-1,-1,1)$.



3) Can you find a triangle in with edges at lattice points $A = (a,b,c)$, $B = (d,e,f)$, $C = (u,v,w)$ such that the sides of the triangle have integer length and the triangle is not contained in any plane parallel to the coordinate planes.

Solution:
Just try or write a little program to help with it. An example is $A = (10,0,3)$, $B = (8,1,1)$, $P = (4,3,5)$. It appears that all these triangles have integer area. We will see that 2 times the area is an integer. The number theoretical problem here is to find integer solutions to a system of nonlinear quadratic equations.