Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points)

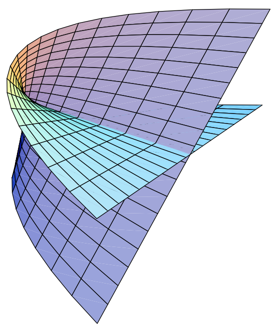
Circle for each of the 20 questions the correct letter. No justifications are needed.

T	F	The length of the sum of two vectors is always the sum of the length of the vectors.
T	F	For any three vectors, $\vec{v} \cdot (\vec{w} + \vec{u}) = \vec{w} \cdot \vec{v} + \vec{u} \cdot \vec{v}$.
T	F	The set of points which satisfy $x^2 + 2x + y^2 - z^2 = 0$ is a cone.
T	F	If P, Q, R are 3 different points in space that don't lie in a line, then $\vec{PQ} \times \vec{RQ}$ is a vector orthogonal to the plane containing P, Q, R.
T	F	The line $\vec{r}(t) = (1 + 2t, 1 + 3t, 1 + 4t)$ hits the plane $2x + 3y + 4z = 9$ at a right angle.
T	F	A surface which is given as $r = \sin(z)$ in cylindrical coordinates stays the same when we rotate it around the y axis.
T	F	For any two vectors, $\vec{v} \times \vec{w} = \vec{w} \times \vec{v}$.
T	F	If $\|\vec{v} \times \vec{w}\| = 0$ for all vectors $\vec{w}$, then $\vec{v} = \vec{0}$.
T	F	If $\vec{u}$ and $\vec{v}$ are orthogonal vectors, then $(\vec{u} \times \vec{v}) \times \vec{u}$ is parallel to $\vec{v}$.
T	F	Every vector contained in the line $\vec{r}(t) = (1 + 2t, 1 + 3t, 1 + 4t)$ is parallel to the vector $(1, 1, 1)$.
T	F	If in spherical coordinates a point is given by $(\rho, \theta, \phi) = (2, \pi/2, \pi/2)$, then its rectangular coordinates are $(x, y, z) = (0, 2, 0)$.
T	F	The set of points which satisfy $x^2 - 2y^2 - 3z^2 = 0$ form an ellipsoid.
T	F	If $\vec{v} \times \vec{w} = (0, 0, 0)$, then $\vec{v} = \vec{w}$.
T	F	The set of points in $\mathbf{R}^3$ which have distance 1 from a line form a cylinder.
T	F	If in rectangular coordinates, a point is given by $(1, 0, 1)$, then its spherical coordinates are $(\rho, \theta, \phi) = (\sqrt{2}, \pi/2, -\pi/2)$.
T	F	In spherical coordinates, the equation $\cos(\theta) = \sin(\theta)$ defines the plane $x - y = 0$.
T	F	For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, we always have $(\vec{a} \times \vec{b}) \cdot \vec{c} = -(\vec{a} \times \vec{c}) \cdot \vec{b}$.
T	F	The set of points in the xy-plane which satisfy $x^2 - y^2 = -1$ is a hyperbola.
T	F	If $\|\vec{v} \times \vec{w}\| = 0$ then $\vec{v} = 0$ or $\vec{w} = 0$.
T	F	Two nonzero vectors are parallel if and only if their cross product is $\vec{0}$.

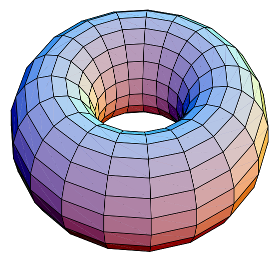
Problem 2) (10 points)

Match the surfaces with their parameterization $\vec{r}(u,v)$ or the implicit description $g(x,y,z)=0$. Note that one of the surfaces is not represented by a formula. No justifications are needed in this problem.

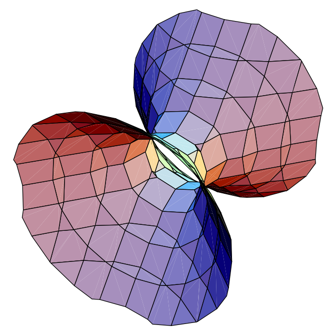
I



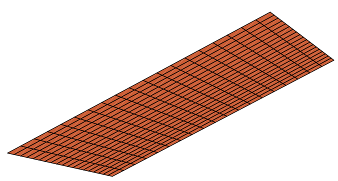
II



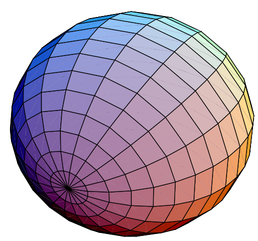
III



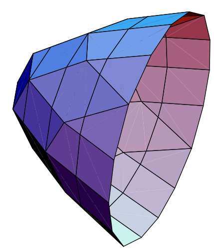
IV



V



VI



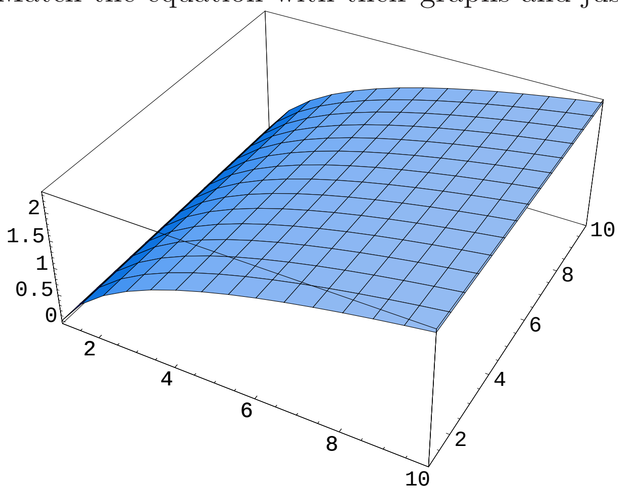
Enter I,II,III,IV,V,VI here	Equation or Parameterization
	$\vec{r}(u,v) = ((1+\sin(u))\cos(v), (1+\sin(u))\sin(v), \cos(u))$
	$\vec{r}(u,v) = (v, v-u, u+v)$
	$\vec{r}(u,v) = (u^2, vu, v)$
	$x^2 - y^2 + z^2 - 1 = 0$
	$\vec{r}(u,v) = (\cos(u)\sin(v), \cos(v), \sin(u)\sin(v))$

Solution:

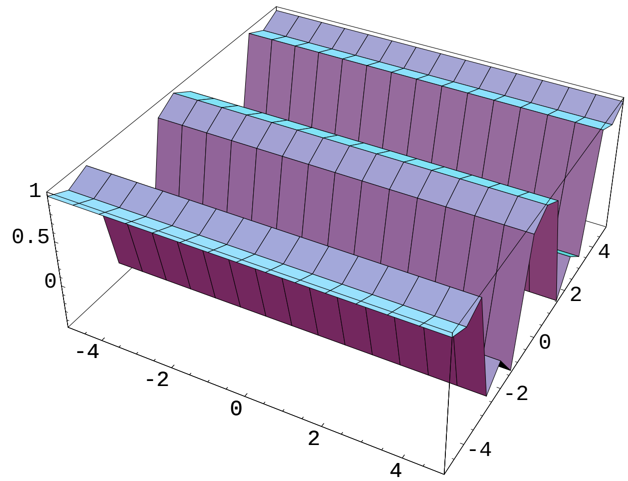
Enter I,II,III,IV,V,VI here	Equation or Parameterization
II	$\vec{r}(u,v) = ((1+\sin(u))\cos(v), (1+\sin(u))\sin(v), \cos(u))$
IV	$\vec{r}(u,v) = (v, v-u, u+v)$
I	$\vec{r}(u,v) = (u^2, vu, v)$
III	$x^2 - y^2 + z^2 - 1 = 0$
V	$\vec{r}(u,v) = (\cos(u)\sin(v), \cos(v), \sin(u)\sin(v))$

Problem 3) (10 points)

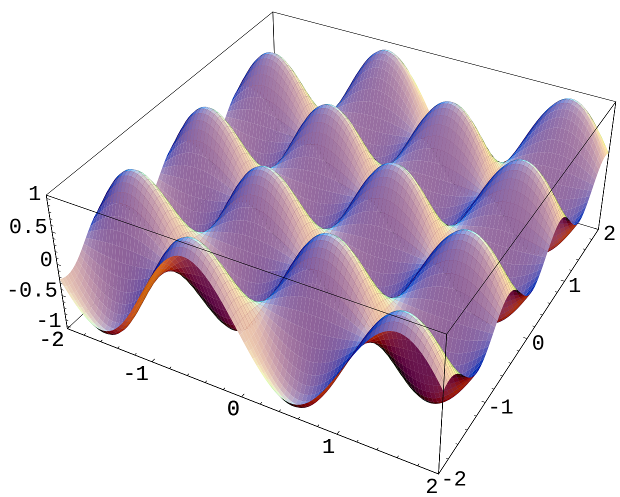
Match the equation with their graphs and justify briefly your choice.



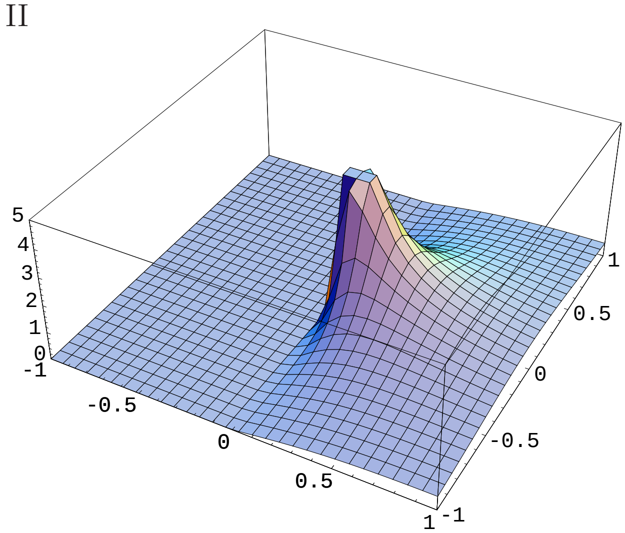
I



II



III



IV

Enter I,II,III,IV here	Equation	Short Justification
	$z = \sin(3x)\cos(5y)$	
	$z = \cos(y^2)$	
	$z = \log(x)$	
	$z = x/(x^2+y^2)$	

Solution:

Enter I,II,III,IV here	Equation	Short Justification
III	$z = \sin(3x)\cos(5y)$	two traces show waves
II	$z = \cos(y^2)$	no x dependence, periodic in y
I	$z = \log(x)$	no y dependence, monotone in x
IV	$z = x/(x^2 + y^2)$	singular at (x,y)=(0,0)

Problem 4) Distances (10 points)

Let L be the line

$$x = 1 + 2t, y = -3t, z = t$$

and let S be the plane $x + y + z = 2$.

a) Verify that L and S have no intersections.

b) Compute the distance between the line L and plane S.

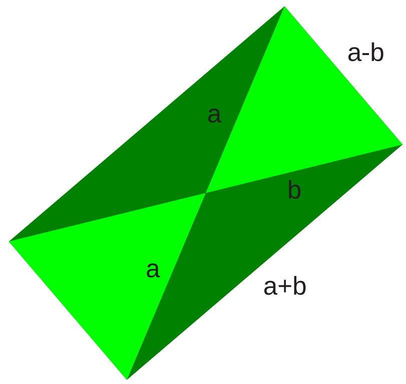
Hint. Just take any point P on the line and compute the distance from the line to the plane.

Solution:
The vector $v = (2, -3, 1)$ is in the line. It is is normal to the normal vector $(1, 1, 1)$ of the plane.
The point $P = (1, 0, 0)$ is on the line. The point $Q = (1, 1, 0)$ is on the plane. The distance is the scalar projection of PQ onto the normal vector $(1, 1, 1)$ which is $(0, 1, 0) \cdot (1, 1, 1)/\sqrt{3} = 1/\sqrt{3}$

Problem 5) (10 points)

Let $\vec{a}$ and $\vec{b}$ be two vectors in $\mathbf{R}^3$. Assume that the length of $\vec{a} \times \vec{b}$ is equal to 10. What is the length of $(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})$?

Solution:



$(\vec{a}+\vec{b})\times(\vec{a}-\vec{b}) = \vec{a}\times\vec{a}-\vec{a}\times\vec{b}+\vec{b}\times\vec{a}-\vec{b}\times\vec{b} = -\vec{a}\times\vec{b}+\vec{b}\times\vec{a} = -2\vec{a}\times\vec{b}$ has length twice the length of $\vec{a}\times\vec{b}$. The answer is $\boxed{20}$. This problem can also be solved geometrically. A single picture is necessary. The paralelogram spanned by (a-b) and (a+b) contains four triangles, each of which is half of the parallelogram spanned by a and b.

Problem 6) (10 points)

Find the distance between the line

$$\vec{r}_1(t) = (t, 2t, -t)$$

and the line which goes through the two points $(1,0,0)$ and $(2,1,1)$.

Solution:
$A = (0,0,0)$ is a point on the first line and $B = (1,0,0)$ is a point on the second line. The vector $\vec{n} = (1,2,-1)\times(1,1,1) = (3,-2,-1)$ is the direction of the vector connecting the closest points.
The distance is $d = \vec{n}\cdot AB/|\vec{n}| = \boxed{3/\sqrt{14}}$.

Problem 7) (10 points)

Given the vectors $v = (1,1,0)$ and $w = (0,0,1)$ and the point $P = (2,4,-2)$. Let Σ be the plane which goes through the origin and contains the vectors v and w.

a) Determine the distance from P to the origin.

Solution:
$\sqrt{4+16+4} = \sqrt{24} = 2\sqrt{6}$.

b) Determine the distance from P to the plane Σ.

Solution:
$\Sigma : x - y = 0$, $n = (1, -1, 0)$. $Q = (0,0,0)$ is a point on the plane. $\vec{PQ} \cdot n/|n| = (2,4,-2) \cdot (1,-1,0)/\sqrt{2} = 2/\sqrt{2} = \sqrt{2}$

Problem 8) (10 points)

a) (5 points) Verify that the triple scalar product has the property $[\vec{u}+\vec{v}, \vec{v}+\vec{w}, \vec{w}+\vec{u}] = 2[\vec{u}, \vec{v}, \vec{w}]$.

b) (5 points) Verify that the triple scalar product $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$ has the property

$$|[\vec{u}, \vec{v}, \vec{w}]| \leq ||\vec{u}|| \cdot ||\vec{v}|| \cdot ||\vec{w}||$$

This is a special case of a famous inquality called **Hadamard inequality**.

Solution:
a) $[u+v, v+w, w+u] = [u,v,w] + [u,v,u] + [u,w,w] + [u,w,u] + [v,v,w] + [v,v,u] + [v,w,w] + [v,w,u]$. Any term, where two parallel vectors appear is zero. So, only $2[u,v,w]$ remains on the right hand side.
b) Build the parallalepiped spanned by u, v, w and note that one can shear it in such a way that it is contained in the box of size $||\vec{u}||$ and $||\vec{v}||$ and $||\vec{w}||$.

Problem 9) (10 points)

a) (3 points) Let S be the surface $g(x,y,z) = x^2 - z - y^2 = 1$. Find a parametrization

$$\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$$

of this surface.

b) (3 points) Write down the parametrization

$$\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$$

of the part of the unit sphere $x^2 + y^2 + z^2 = 1$ which satisfies $z \geq \sqrt{3}/2$ and also indicate the domain R of the parametrization.

c) (4 points) Let S be the surface given in cylindrical coordinates as $r = 2 + \sin(z)$. Find a parameterization

$$\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$$

of the surface.

Solution:
a) this is a graph. Write $r(u, v) = (u, v, u^2 - v^2 - 1)$.
b) $r(u, v) = (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$ with $0 \leq u \leq 2\pi$ and $0 \leq v \leq \pi/6$. c) The distance to the z-axis is $2 + \sin(z)$. We take the rotation angle θ as a second parameter. Therefore $\vec{r}(\theta, z) = ((2 + \sin(z))\cos(\theta), (2 + \sin(z))\sin(\theta), z)$.