7/14/2005 FIRST MIDTERM **Maths 21a, O. Knill, Summer 2005**

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) No justifications are needed.

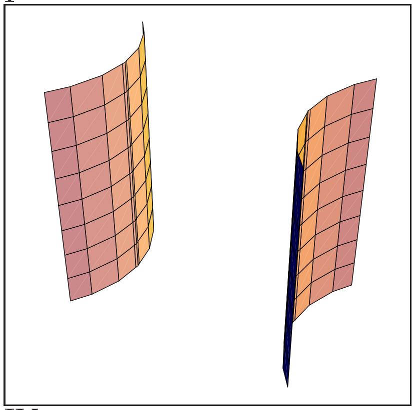
T	F	The vector $\vec{v} = (1, 5, 7)$ is perpendicular to the plane $x + 5y + 7z = 100$.
T	**F**	For any three vectors $\vec{u}, \vec{v}, \vec{w}$, the identity $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{w} \times \vec{v}) \cdot \vec{u}$ holds.
T	**F**	With $\vec{i} = (1, 0, 0), \vec{j} = (0, 1, 0), \vec{k} = (0, 0, 1)$, the formula $(\vec{i} \times \vec{k}) \times (\vec{j} \times \vec{k}) = \vec{0}$ holds.
T	**F**	The vectors $\vec{u} = (3, -2, 1)$ and $\vec{v} = (-6, 4, 2)$ are parallel.
T	**F**	The set of points which have distance 1 from the xy-plane is a single plane parallel to the xy-plane.
T	F	If $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$ are perpendicular, then the vectors $\vec{u}$ and $\vec{v}$ have the same length.
T	F	The two vectors $(2, 3, 0)$ and $(6, -4, 5)$ are orthogonal.
T	**F**	For any two vectors $\vec{v}, \vec{w}$, one has $\lvert\vec{v} + \vec{w}\rvert^2 = \lvert\vec{v}\rvert^2 + \lvert\vec{w}\rvert^2$.
T	F	The surface $x^2 - y^2 + z^2 = 1$ is called a one-sheeted hyperboloid.
T	F	The set of points which have distance 1 from the x-axis is a cylinder.
T	F	If in spherical coordinates a point is given by $(\rho, \theta, \phi) = (2, \pi/2, \pi/2)$, then its Euclidean coordinates is $(x, y, z) = (0, 2, 0)$.
T	F	If the distance between two lines is zero, then the two lines belong to the same plane.
T	F	A surface which is given as $r = 2 + \sin(z)$ in cylindrical coordinates stays the same when we rotate it around the z axis.
T	**F**	The length of the difference $\vec{v} - \vec{w}$ of two parallel vectors $\vec{v}, \vec{w}$ is the difference $\lvert\vec{v}\rvert - \lvert\vec{w}\rvert$ of the lengths of the vectors.
T	**F**	The volume of a parallelepiped spanned by $(1, 0, 0), (0, 1, 0)$ and $(1, 1, 1)$ is equal to $1/3$.
T	F	The equation $x^2 - z^2 = y$ describes a hyperbolic paraboloid.
T	F	In spherical coordinates, the equation $\cos(\theta) = \sin(\theta)$ is the plane $x - y = 0$.
T	**F**	If $\lvert\vec{v} \times \vec{w}\rvert = 0$ then $\vec{v} = 0$ or $\vec{w} = 0$.
T	F	The vector projection of the vector $(1, 1, 1)$ onto the vector $(0, 2, 0)$ is $(0, 1, 0)$.
T	**F**	To every point on the unit sphere corresponds exactly one spherical coordinate (ρ, θ, ϕ) with $\rho = 1, 0 \le \theta < 2\pi, 0 \le \phi \le \pi$.

Total

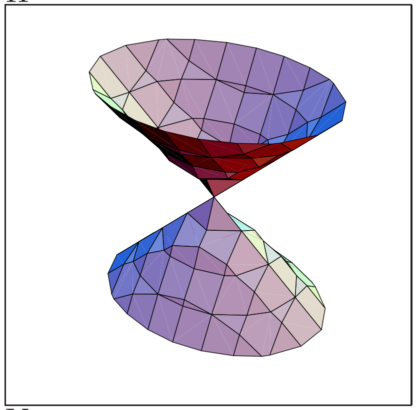
Problem 2) (10 points)

Match the equations with the surfaces.

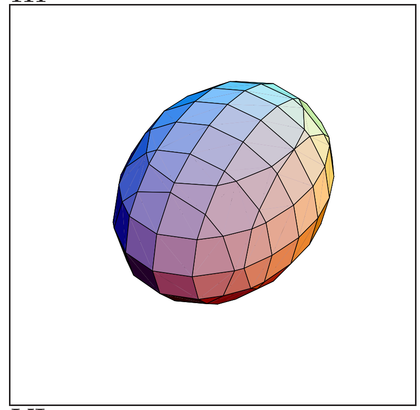
I



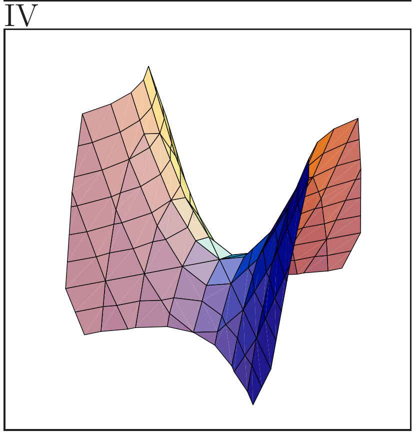
II



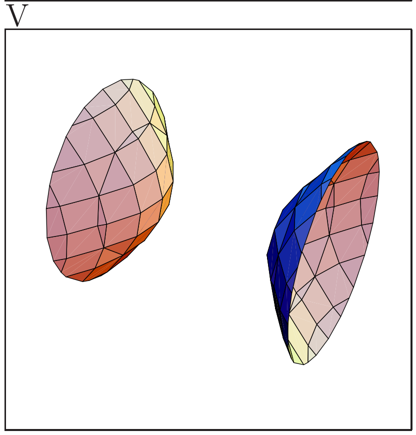
III



IV



V



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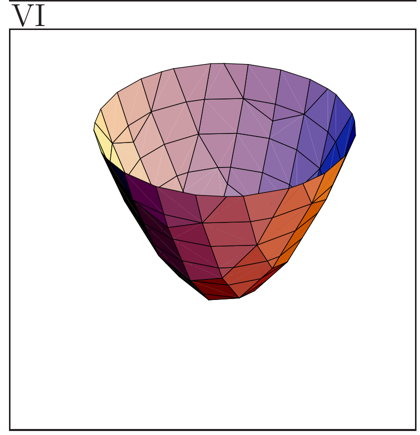


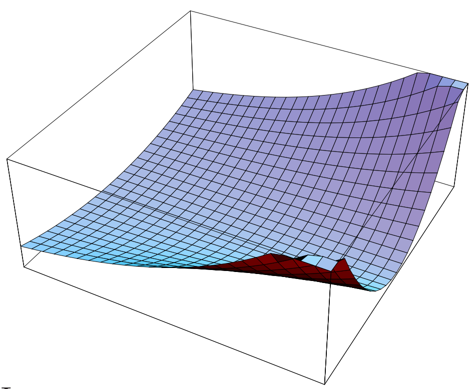
Enter I,II,III,IV,V,VI here	Equation
	$x^2 - y^2 - z^2 = 1$
	$x^2 + 2y^2 = z^2$
	$2x^2 + y^2 + 2z^2 = 1$
	$x^2 - y^2 = 5$
	$x^2 - y^2 - z = 1$
	$x^2 + y^2 - z = 1$

Solution:

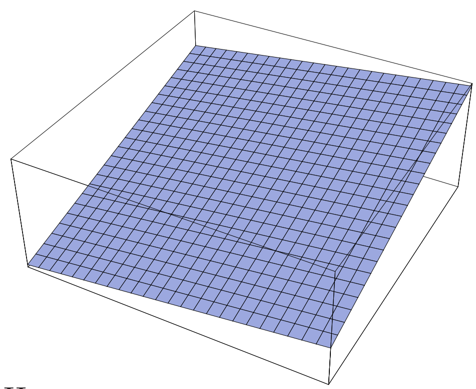
Enter I,II,III,IV,V,VI here	Equation
V	$x^2 - y^2 - z^2 = 1$
II	$x^2 + 2y^2 = z^2$
III	$2x^2 + y^2 + 2z^2 = 1$
I	$x^2 - y^2 = 5$
IV	$x^2 - y^2 - z = 1$
VI	$x^2 + y^2 - z = 1$

Problem 3) (10 points)

Match the equation with their graphs. No justifications are needed.



I



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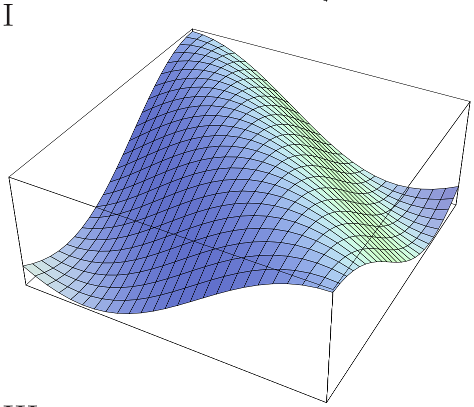

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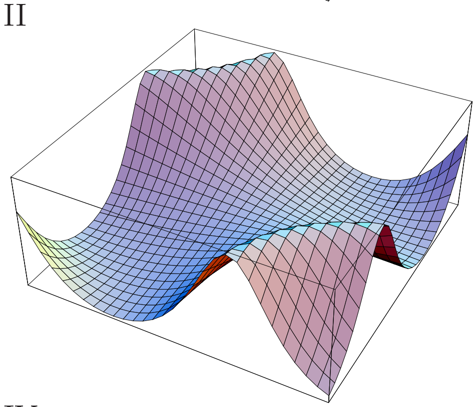

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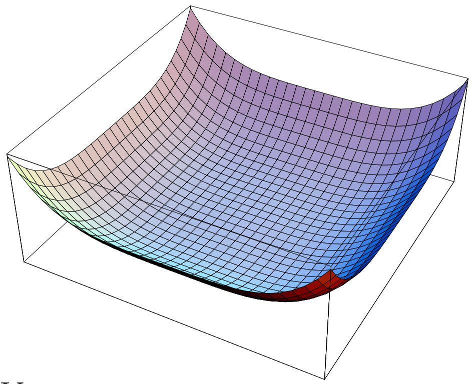

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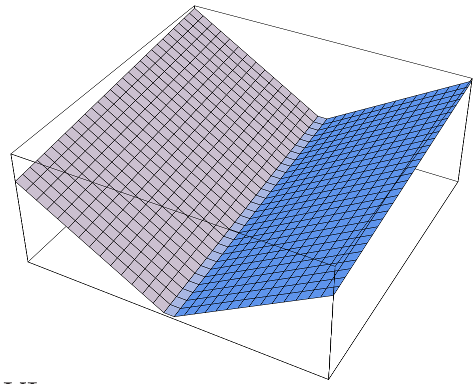


VI

Enter I,II,III,IV,V or VI here	Equation
	$z = \|x\| + y$
	$z = y^2 e^x$
	$z = \cos(x+y)$
	$z = x + y$
	$z = x/(2+\sin(xy))$
	$z = x^4 + y^4$

Solution:

Enter I,II,III,IV,V or VI here	Equation
VI	$z = \|x\| + y$
I	$z = y^2 e^x$
III	$z = \cos(x+y)$
II	$z = x + y$
V	$z = x/(2+\sin(xy))$
VI	$z = x^4 + y^4$

Problem 4) (10 points)

Compute:

a) $(4,5,1) \cdot (1,-1,1)$

b) $(1,1,3) \times (1,1,1)$

c) $(2,1,3) \cdot ((3,4,5) \times (1,1,3))$.

d) $\vec{\text{proj}}_{(1,0,0)}(7,3,2)$

e) $\text{comp}_{(0,1,0)}(7,3,2)$

Solution:
a) 0.
b) $(-2,2,0)$.
c) $((3,4,5)\times(2,1,3))=(7,-4,-1)$ and the result is 7.
d) This is a vector projection: $(7,0,0)$.
e) This is a scalar projection: 3.

Problem 5) (10 points)

Find the distance between the point $P=(1,0,-1)$ and the line which contains the points $A=(1,1,1)$ and $B=(0,2,1)$.
To do so:

a) (4 points) Find first a parametrization $\vec{r}(t)=Q+t\vec{v}$ of the line.

Solution:
$\vec{v}=\vec{AB}=((0,2,1)-(1,1,1))=(-1,1,0)$ is in the line. ' A parametrization is $\vec{r}(t)=(1,0,-1)+t(-1,1,0)$. We can write it as $\vec{r}(t)+(1-t,t,-1)$.

b) (6 points) Now find the distance.

Solution:
Use the distance formula: $|\vec{AP}\times\vec{v}|/|\vec{v}|=|(0,-1,-2)\times(-1,1,0)|\|(-1,1,0)|=\boxed{3/\sqrt{2}}$.

Problem 6) (10 points)

a) (4 points) Find a parameterization of the line of intersection of the planes $3x-2y+z=7$ and $x+2y+3z=-3$.

Hint. Use the fact that the line goes through the point $P=(1,-2,0)$.

b) (3 points) Find the symmetric equations

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

representing that line.

c) (3 points) Give a formula for the angle α between the two planes. You do not have to evaluate the result numerically and leaving square roots is ok.

Hint. The angle between two planes is defined as the angle between the two normal vectors of the planes.

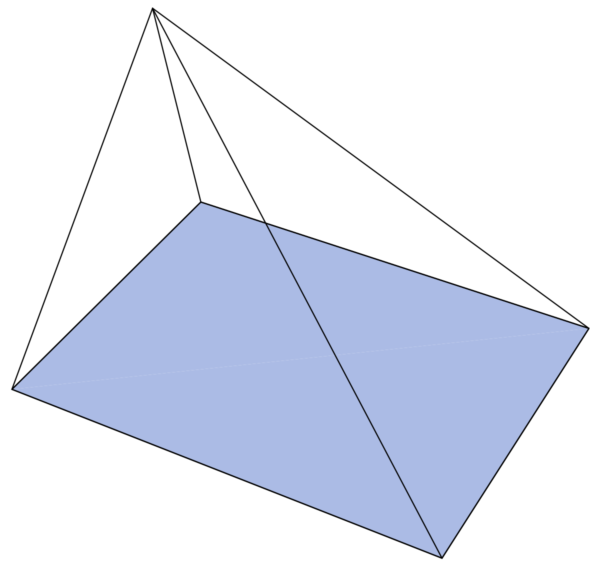
Solution:
a) The line of intersection is perpendicular to both planes. It contains the vector $(3, -2, 1) \times (1, 2, 3) = 8(-1, -1, 1)$. A parameterization is $\boxed{\vec{r}(t) = (1, -2, 0) + t(-1, -1, 1)}$.
b) If a line contains the point (x_0, y_0, z_0) and a vector (a, b, c), then the symmetric equation is $(x - x_0)/a = (y - y_0)/b = (z - z_0)/c$. In our case, where $(x_0, y_0, z_0) = (1, -2, 0)$ and $(a, b, c) = (-1, -1, 1)$, the symmetric equations are $\boxed{x - 1 = y + 2 = -z}$.
c) $\cos(\alpha) = |(3, -2, 1) \cdot (1, 2, 3)|/(|(3, 2, 1)||(1, 2, 3)|) = 2/14 = 1/7$. So $\alpha = \arccos(1/7)$.

Problem 7) (10 points)

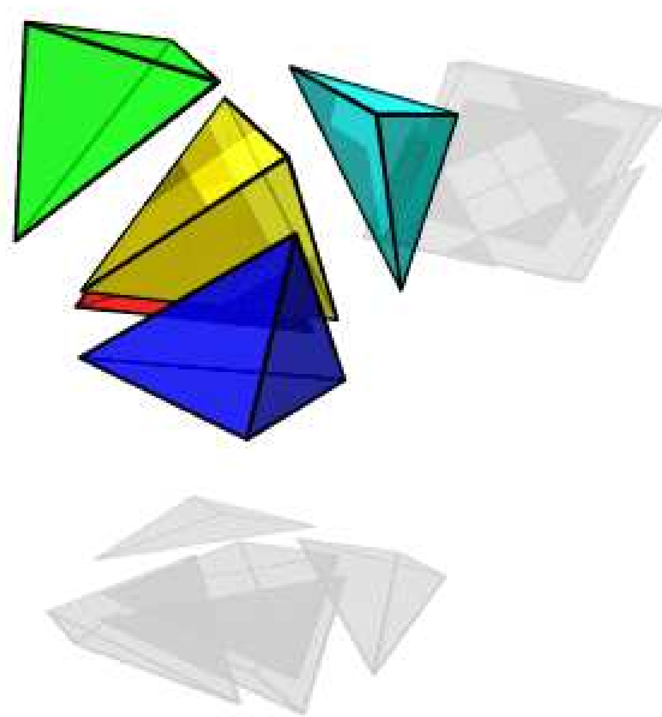
a) (5 points) Find the area of the parallelogram $PQSR$ with corners

$$P = (0, 0, 0), Q = (1, 1, 1), R = (1, 1, 0), S = (2, 2, 1) \ .$$

b) (5 points) Find the volume of the pyramid which has as the base the parallelogram $PQRS$ and has a fifth vertex at $T = (3, 4, 3)$.



Solution:
a) The area is $|(Q-P)\times(R-P)| = \sqrt{2}$.
b) The distance $h = |w \cdot (u \times v)|/|w|$ of T to the plane containing the points P, Q, S, R and use the volume formula $V = Ah/3$, where $A = |u \times v|$ is the area of the parallelogram. The volume of the pyramid is $\boxed{1/3}$. Remark. The parallelepiped spanned by $\vec{u} = (1,1,1), \vec{v} = (1,1,0)$ and $\vec{w} = (2,2,1)$ has volume 1. The pyramid has volume 1/3 of this volume because one can chop the pyramid into two tetraedra of the same volume and each tetrahedron has volume 1/6. One can take 4 tetrahedra spanned by u, v, w as well as one of double volume spanned by $(u+v, u+w, v+w)$ to build the parallelepiped. The volume of the pyramid is therefore $\boxed{1/3}$.
The following picture shows that a tetrahedron spanned by vectors u, v, w has a volume 1/6'th of the volume of the parallelepiped:



Problem 8) (10 points)

In this problem, it is enough to describe the surface with words.

a) (3) Identify the surface whose equation is given in spherical coordinates as $\phi = \pi/6$.

b) (3) Identify the surface whose equation is given in spherical coordinates as $\theta = \pi/2$.

c) (2) Identify the surface, whose equation is given in cylindrical coordinates by $z^2 = r$.

d) (2) Identify the surface, whose equation is given in cylindrical coordinates as $r\cos(\theta) = 1$

Solution:
a) A **half cone**. $x^2 + y^2 = z^2$ with $z \geq 0$.
b) A **half plane** contained in the plane $x = 0$ and containing the positive y axes.
c) This surface appeared in the homework. It is a "concave cone-type" surface of revolution. The trace can be obtained by drawing $z = \pm\sqrt{r}$. If you spin this graph wround the z axes, you obtain the surface.
d) This is the **plane** $x = 1$. Just note that $r\cos(\theta) = x$ in cylindrical coordinates.

Problem 9) (10 points)

Remember that a parameterization of a surface describes the points (x, y, z) of the surface in the form $\vec{r}(u, v) = (x, y, z) = (x(u, v), y(u, v), z(u, v))$. What surfaces do the following parameterizations represent? Find in each case an implicit equation of the form $g(x, y, z) = c$ which is equivalent.

a) (3) $\vec{r}(u, v) = (\cos(u), \sin(u), v)$

b) (3) $\vec{r}(u, v) = (u + v, v - u, u + 2v)$

c) (2) $\vec{r}(u, v) = (v\cos(u), v\sin(u), v)$

d) (2) $\vec{r}(u, v) = (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$.

Solution:
a) The cylinder $x^2 + y^2 = 1$.
b) A plane containing the vectors $(1, -1, 1)$ and $(1, 1, 2)$. To get the equation of the plane, take the cross product. Because the plane passes through the origion, we have $-3x - y + 2z = 0$.
c) A cone $x^2 + y^2 = z^2$.
d) The unit sphere: $x^2 + y^2 + z^2 = 1$.