This is part 3 (of 3) of the weekly homework. It is due Monday August 16 at the review.

SUMMARY.

- $\mathrm{div}(F)(x,y,z) = P_x + Q_y + R_z$ **divergence** of the vector field $F = (P,Q,R)$. Is a scalar field measuring the how much field is "produced" at a point (x,y,z). It measures the rate of "expansion" at (x,y,z).
- $\int\int\int_E \mathrm{div}(F)\ dV = \int\int_S F \cdot dS$ **divergence theorem** also called **Gauss theorem**. The surface S bounds a solid E in space. It is oriented so that the normal vector points away from the solid.

Homework Problems

1) (4 points) Compute using the divergence theorem the flux of the vector field $F(x,y,z) = (3y, xy, 2yz)$ through the unit cube $[0,1] \times [0,1] \times [0,1]$.

Optional. If you have time, check your result by computing the flux directly. You would have to compute 6 flux integrals, one through each face.

Solution:
The cube consists of 6 faces. The flux through the face $x = 1$ is $\int_0^1 \int_0^1 3y\ dydz = 3/2$.
The flux through the face $x = 0$ is $-3/2$.
The flux through the face $y = 1$ is $\int_0^1 \int_0^1 x\ dxdz = 1/2$.
The flux through the face $y = 0$ is 0.
The flux through the face $z = 1$ is $\int_0^1 \int_0^1 2y\ dxdy = 1$.
The flux through the face $z = 0$ is $\int_0^1 \int_0^1 0\ dxdy = 0$.
The sum of all these fluxes is $3/2$.
The divergence of F is $x + 2y$. Integrating this over the unit cube gives $1/2 + 1 = 3/2$.

2) (4 points) Find the flux of the vector field $F(x,y,z) = (xy, yz, zx)$ through the solid cylinder $x^2 + y^2 \leq 1$, $0 \leq z \leq 1$.

Optional. If you have time, check your result by computing the flux directly. You have to compute the flux through three surfaces, the top, the bottom and the mantle of the cylinder.

Solution:
The divergence is $(y + z + x)$. Integrated over the cylinder gives $\int_0^1 z\ dz\pi = \pi/2$.
The flux of the vector field through the bottom is 0 because there the vector field has the form (*,*,0)$andthenormalvectoris$(0,0,-1). The flux integral over the top is $\int\int_R x\ dxdy$ where R is the unit disc, which is zero. To compute the flux integral over the boundary of the cylinder, parametrize the cylinder as $\vec{r}(\theta, z) = (\cos(\theta), \sin(\theta), z)$. We have $F(\vec{r}(u,v)) = (\cos(\theta \sin(\theta), \sin(\theta)z, \cos(\theta)z)$ and $r_u \times r_v = (\cos(\theta), \sin(\theta), 0)$. The flux integral is $\int_0^1 \int_0^{2\pi} \cos^2(\theta)\sin(\theta) + \sin^2(\theta)z\ d\theta dz = \pi/2$.

3) (4 points) Use the divergence theorem to calculate the flux of $F(x,y,z) = (x^3, y^3, z^3)$ through the sphere $x^2 + y^2 + z^2 = 1$.

Solution:
The divergence of F is $3x^2 + 3y^2 + 3z^2 = 3\rho^2$. By the divergence theorem, we have to integrate this over the interior of the sphere. This is done best in spherical coordinates. The integral is

$$\int_0^{2\pi} \int_0^{\pi} 3\rho^4 \sin(\phi)\ d\rho d\phi\ d\theta = 12\pi/5\ .$$

4) (4 points)
a) Verify that $\mathrm{div}(E) = 0$ away from the origin if E is the electric field $E(\vec{x}) = \vec{x}/|\vec{x}|^3$.
b) An electric charge at 0 generates the field E as in a). What is the flux of E through the unit sphere S?
c) What is the flux of E through any other sphere containing the origin $(0,0,0)$ inside?
d) What is the flux of the electric field $F(\vec{x}) = \sum_{i=1}^n E(\vec{x} - \vec{x}_i)$ through S generated by n charges located at points $\vec{x}_i$ inside S?

Solution:
a) The vector field is $(P,Q,R) = (x(x^2+y^2+z^2)^{-1/2}, y(x^2+y^2+z^2)^{-1/2}, z(x^2+y^2+z^2)^{-1/2})$.
$P_x = (x^2+y^2+z^2)^{-3/2} - \frac{3}{2}2x^2(x^2+y^2+z^2)^{-5/2}$.
$Q_y = (x^2+y^2+z^2)^{-3/2} - \frac{3}{2}2y^2(x^2+y^2+z^2)^{-5/2}$.
$R_z = (x^2+y^2+z^2)^{-3/2} - \frac{3}{2}2z^2(x^2+y^2+z^2)^{-5/2}$.
$P_x + Q_y + R_z = 0$.
b) The flux is 4π. First verify that for a sphere which is centered at the charge. For any other sphere, the flux is the same.
c) The flux through any other sphere which does not contain a charge is zero by the divergence theorem.
d) The flux is $4\pi n$. Each charge generates a flux of 4π.

5) (4 points) Find $\int\int_S F \cdot dS$, where $F(x,y,z) = (x,y,z)$ and S is the outwardly oriented surface obtained by removing the cube $[1,2] \times [1,2] \times [1,2]$ from the cube $[0,2] \times [0,2] \times [0,2]$.

Solution:
The divergence is 3. By the divergence theorem, the result is 3 times the area of the solid E which is $(8-1)3 = 21$.

Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

1) We show that Green's theorem in the plane is equivalent to Gauss theorem in the plane:

a) Let $G(x,y) = (Q,-P)$ be the vector field orthogonal to $F(x,y) = (P,Q)$. Show that $\mathrm{div}(G) = \mathrm{curl}(F)$.

b) Show that the line integral $\int_C F\ dr$ along a curve C is the same as the flux integral $\int_C G \cdot dn$, where dn is a vector perpendicular to the curve with the same length as $r'(t)dt$.

2) Formulate the divergence theorem in arbitrary dimensions.

3) The human civilisation managed in the year 2200 to build a colony on Mars. In the year 2500 AC, after more than 100 years of work done by robots, one managed to clear an inner core of mars in order to build a safe data storage and computing facility center which is shielded from

radiation. The planet is now a shell with outer radius R and inner radius r. Assuming that the density of the Mars material is constant everywhere, how does the gravitational force behave inside this "hollow Mars"?

Image credit: http://www.astrosurf.com/lombry/colonisation.htm