

This is part 3 (of 3) of the weekly homework. It is due August 9'th in class.

## SUMMARY.

- $\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t) dt$  **line integral** of  $F$  along curve  $C : t \mapsto r(t)$ .
- Example:  $C : r(t) = (\cos(t), \sin(t)), t \in [0, 2\pi]$  (circle),  $F(x, y) = (-y, x)$ .  $\int_C F \cdot dr = \int_0^{2\pi} F(r(t)) \cdot r'(t) dt = \int_0^{2\pi} (-\sin(t), \cos(t)) \cdot (-\sin(t), \cos(t)) dt = \int_0^{2\pi} 1 dt = 2\pi$ .
- We say  $F$  is **conservative** or  $F$  is a **gradient field** if  $F(x, y, z) = \nabla f(x, y, z)$ .
- $\int_C \nabla f \cdot dr = f(r(b)) - f(r(a))$  **fundamental theorem of line integrals**: by chain rule:  $\int_a^b \nabla f(r(t)) \cdot r'(t) dt = \int_a^b \frac{d}{dt} f(r(t)) dt$ . Apply the fundamental theorem of calculus.
- If  $F$  is conservative in the plane then the line integrals do not depend on the path.
- If  $F$  is conservative in the plane and  $C$  is a closed curve, then  $\int_C F \cdot dr = 0$ .

## Homework Problems

- 1) (4 points) Find a closed curve  $C : \vec{r}(t)$  for which the vector field  $F(x, y) = (P(x, y), Q(x, y)) = (xy, x^2)$  satisfies  $\int_C F(r(t)) \cdot r'(t) dt \neq 0$ .

**Solution:**

We knew already that, because  $Q_x = 2x$  and  $P_y = x$ , the field can not be a gradient field  $(P, Q) = (f_x, f_y)$ .

- 2) (4 points) Let  $C$  be the circle  $x^2 + y^2 = 16$  and  $F(x, y) = (x, y^4)$ . Calculate the line integral  $\int_C F \cdot dr$ .

**Solution:**

Actually, this is a conservative field because  $F(x, y) = \nabla f(x, y)$  with  $f(x, y) = x^2/2 + y^5/5$  so that the line integral is zero.

However, we can also compute the line integral:  $\vec{r}(t) = (4 \cos(t), 4 \sin(t))$  parametrizes the circle.  $\vec{r}'(t) = (-4 \sin(t), 4 \cos(t))$  and  $F(r(t)) = (4 \cos(t), 256 \sin^4(t))$  so that  $\int_0^{2\pi} F(r(t)) \cdot r'(t) dt = \int_0^{2\pi} (-16 \sin(t) \cos(t) + 1024 \sin^4(t) \cos(t)) dt = (8 \sin^2(t) + 1024 \sin^5(t)/5t)|_0^{2\pi} = 0$ .

- 3) (4 points) Let  $C$  be the space curve  $\vec{r}(t) = (\cos(t), \sin(t), t)$  for  $t \in [0, 1]$  and let  $F(x, y, z) = (y, x, 5)$ . Calculate the line integral  $\int_C F \cdot dr$ .

**Solution:**

$\int_C f \cdot dr = \int_0^1 (\sin(t), \cos(t), 5) \cdot (-\sin(t), \cos(t), 1) dt = \int_0^1 (\cos(2t) + 5) dt = \sin(2)/2 + 5$ .

- 4) (4 points) Find the work done by the force field  $F(x, y) = (x \sin(y), y)$  on a particle that moves along the parabola  $y = x^2$  from  $(-1, 1)$  to  $(2, 4)$ .

**Solution:**

We parametrize the curve  $C$  by  $r(t) = (t, t^2)$  so that  $r'(t) = (1, 2t)$ .

$W = \int_C F \cdot dr = \int_{-1}^2 (t \sin(t^2), t^2) \cdot (1, 2t) dt = \int_{-1}^2 (t \sin(t^2) + 2t^3) dt = \cos(1)/2 - \cos(4)/2 + 15/2$ .

- 5) (4 points) Let  $F$  be the vector field  $F(x, y) = (-y, x)/2$ . Compute the line integral of  $F$  along an ellipse  $\vec{r}(t) = (a \cos(t), b \sin(t))$  with width  $2a$  and height  $2b$ . The result should depend on  $a$  and  $b$ .

**Solution:**

The velocity is  $\vec{r}'(t) = (-a \sin(t), b \cos(t))$  and  $F(\vec{r}(t)) = (-b \sin(t), a \cos(t))/2$  so that  $F(\vec{r}(t)) \cdot \vec{r}'(t) = ab/2$ . If we integrate this from 0 to  $2\pi$  we get the result  $\pi ab$ .

## Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

- 1) Consider a O shaped pipe which is filled only on the right side with water. A wooden ball falls on the right hand side in the air and moves up in the water. Why does this "perpetual motion machine" not work?



- 2) What is wrong with the Escher pictures with the stair in which people always walk down or with the waterfall. The figures suggests the existence of a force field which is not conservative. How do the Escher pictures "work"?

