This is part 2 (of 2) of the weekly homework. It is due August 2 at the beginning of class.

SUMMARY. $dA = dxdy = rdrd\theta$ area element.

- $\int\int_R f(x,y)dxdy = \int_\alpha^\beta \int_a^b f(r\cos(\theta), r\sin(\theta))\; rdrd\theta$ **integral in polar coordinates**.
- $\int\int_R f(x,y)\; dxdy / \int\int_R 1\; dxdy$ is the **average value** of f on R.
- A curve $\vec{r}(t) = (f(t)\cos(t), f(t)\sin(t))$ can in polar coordinates (r,θ) be given as $r(\theta) = f(\theta)$.
- A vector valued function $\vec{r}(u,v)$ defines a **parametric surface** defined on a region R. It has the **surface area** $\int\int_R |\vec{r}_u(u,v) \times \vec{r}_v(u,v)|\; dudv$.

Homework Problems

1) (4 points) Integrate $f(x,y) = x^2$ over the unit disc $\{x^2 + y^2 \leq 1\}$ in two ways, first using Cartesian coordinates, then using polar coordinates.

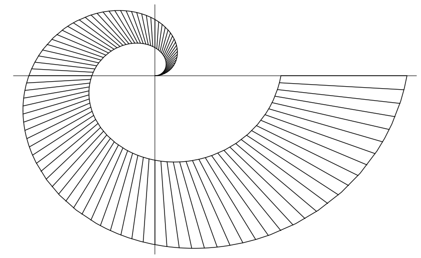
Solution:
The integral in Cartesian coordinates goes less smooth because we have to compute a 1D integral with partial integration: $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2\; dydx = \int_{-1}^1 2x^2\sqrt{1-x^2}\; dx = \pi/4$.
The integral in polar coordinates is easier to get: using the substitution $x = \cos(u)$ we obtain

$$\int_0^{2\pi} \int_0^1 r^2\cos(\theta)^2 r\; drd\theta = (1/4)\pi\;.$$

2) (4 points) Find $\int\int_R (x^2 + y^2)^{10}\; dA$, where R is the part of the unit disc $\{x^2 + y^2 \leq 1\}$ for which $y > x$.

Solution:
Easy in polar coordinates: $\int_0^1 \int_{\pi/4}^{5\pi/4} r^{21} d\theta dr = \pi/22$. This integral would be quite terrible to solve in Cartesian coordinates.

3) (4 points) What is the area of the region which is bounded by three curves, first by the polar curve $r(\theta) = \theta$ with $\theta \in [0, 2\pi]$, second by the polar curve $r(\theta) = 2\theta$ with $\theta \in [0, 2\pi]$ and third by the positive x-axis.



Solution:
$\int_0^{2\pi} \int_\theta^{2\theta} r\; drd\theta = \int_0^{2\pi} (2\theta)^2/2 - (\theta)^2/2\; d\theta = (3/2)\int_0^{2\pi} \theta^2\; d\theta = (3/2)(2\pi)^3/3 = 4\pi^3$.

4) (4 points) Find the average value of $f(x,y) = x^2 + y^2$ on the annulus $1 \leq |(x,y)| \leq 2$.

Solution:
The integral of f over the annulus is $2\pi \int_1^2 r^3\; drd\theta = 2\pi(16-1)/4$. The area is $2\pi \int_1^2 r\; dr = 2\pi(4-1)/2$. The average is $(15/4)/(3/2) = 5/2$.

5) (4 points) Find the surface area of the paraboloid $x = y^2 + z^2$ that lies inside the cylinder $y^2 + z^2 = 9$.

Solution:
We use polar coordinates in the yz-plane. The paraboloid is parametrized by $(u,v) \mapsto (v, v^2\cos(u), v^2\sin(u))$ and the surface integral $\int_0^3 \int_0^{2\pi} |\vec{r}_u \times \vec{r}_v|\; dudv$ is equal to $\int_0^3 \int_0^{2\pi} v\sqrt{1+4v^2}\; dudv = 2\pi \int_0^3 v\sqrt{1+4v^2}\; dv = \pi(37^{3/2} - 1)/6$.

Challenge Problems

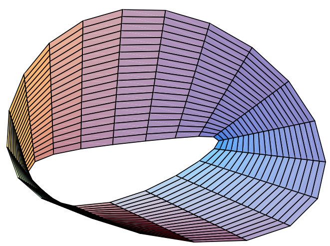
(Solutions to these problems are **not** turned in with the homework.)

1) The M"obius strip is a surface which has only one side. It is parametrized as $(1 + (v - 1/2)\cos(u/2))\cos(u), (1 + (v - 1/2)\sin(u/2))\sin(u), (v - 1/2)\sin(u/2))$. What surface do you compute with the integral

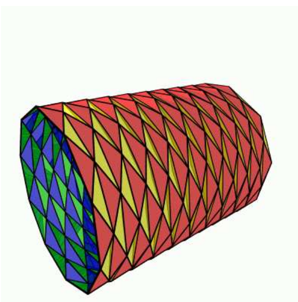
$$\int_0^{2\pi} \int_{-1}^{1} |\vec{r}_u(u,v) \times \vec{r}_v(u,v)| \, dudv \; ?$$

What surface do you compute with the integral

$$\int_0^{4\pi} \int_{-1}^{1} |\vec{r}_u(u,v) \times \vec{r}_v(u,v)| \, dudv \; ?$$



2) In class, you have seen a surface which encloses a finite volume and has infinite surface area. Can you construct for any constant M like $M = 10^{100} cm^2$ a surface inside the unit ball such that the surface area is bigger than M? The picture below should be a hint.



3) Calculate $\int \int_{\mathbf{R}^2} e^{-x^2-y^2} \, dxdy$ and use this to calculate the integral $\int_{-\infty}^{\infty} e^{-x^2} \, dx$.

Hint. The function $f(x) = e^{-x^2}$ is known to have no anti-derivative which can be expressed with "known functions" like $\exp, \log, \sin$ etc. You can nevertheless find a closed solution for the definite integral $\int_{-\infty}^{\infty} e^{-x^2} \, dx$.

4) Find the area of the region shaded in the picture. The region is bounded by the polar curves $r(\theta) = 2\theta$ with $\theta \in [0, 6\pi]$ and $r(\theta) = 3\theta$ with $\theta \in [0, 4\pi]$.