7/26/2005 DOUBLE INTEGRALS **Maths21a** This is part 1 (of 3) of the weekly homework. It is due August 2 at the beginning of class.

SUMMARY. $dA = dxdy$ is called **area element**.

- $\int\int_R f\ dA = \int_a^b \int_c^d f(x,y)\ dydx$ is called a **double integral** over a rectangle R.
- $\int\int_R f\ dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\ dydx$ double integral over a **type I region**.
- $\int\int_R f\ dA = \int_a^b \int_{h_1(y)}^{h_2(y)} f(x,y)\ dxdy$ double integral over a **type II region**.
- $A(R) = \int\int_R 1\ dA$ is called the **area** of R.
- $\frac{1}{A(R)} \int\int_R f\ dA$ is called the **average value** or the **mean** of f on R.
- For $f \geq 0$, the integral $\int_R f\ dA$ is the volume of the solid over R bounded below by the xy-plane and bounded above by the graph of f.

Homework Problems

1) (4 points) Calculate the iterated integral $\int_1^4 \int_0^2 (2x - \sqrt{y})\ dxdy$. Can you interpret it as a volume of a solid? If not, can you express the result using two volumes?

Solution:
Start with the inner integral $\int_0^2 (2x - \sqrt{y})\ dx = 4 - 2\sqrt{y}$. Integrating this from 1 to 4 gives $\boxed{16/3}$.

2) (4 points) Find the area of the region

$$R = \{(x,y) \mid 0 \leq x \leq 2\pi, \sin(x) - 1 \leq y \leq \cos(x) + 2\}$$

and use it to compute the average value of $f(x,y) = y$ over that region.

Remark. You will use here the integral $\int_0^{2\pi} \sin^2(x)\ dx$ treated in class.

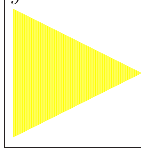
Solution:
In this problem, it helps to see that $\int_0^{2\pi} \cos(x)\ dx = 0$ and $\int_0^{2\pi} \cos^2(x)\ dx = \pi$ and the same for sin.
The area is $A = \int_0^{2\pi} \int_{\sin(x)-1}^{\cos(x)+2} 1\ dxdy = \int_0^{2\pi} (\cos(x)+2) - (\sin(x)-1)\ dx = 6\pi$. The average value is $\int_0^{2\pi} \int_{\sin(x)-1}^{\cos(x)+2} y\ dxdy/A = \int_0^{2\pi} (\cos(x)+2)^2 - (\sin(x)-1)^2\ dx/A = (4-1)\pi/(2\pi) = 3\pi/(6\pi) = \boxed{1/2}$.

3) (4 points) Find the volume of the solid lying under the paraboloid $z = x^2 + y^2$ and above the rectangle $R = [-2,2] \times [-3,3] = \{(x,y) \mid -2 \leq x \leq 2, -3 \leq y \leq 3\}$.

Solution:
We have to compute the double integral of $f(x,y) = x^2 + y^2$ over R. The inner integral is $\int_{-3}^{3} (x^2 + y^2)\ dx = 18 + 6y^2$ so that $\int_{-2}^{2} \int_{-3}^{3} (x^2 + y^2)\ dydx = \int_{-2}^{2} (18 + 6x^2)\ dx = 104$.

4) (4 points) Calculate the iterated integral $\int_0^1 \int_x^{2-x} (x^2 - y)\ dydx$. Sketch the corresponding type I region. Write this integral as integral over a type II region and compute the integral again.

Solution:
$\int_0^1 \int_x^{2-x} (x^2 - y)\ dydx = -5/6$. The region is a triangle bound by the lines $y = x$, the line $y = 2 - x$ and the y axis. The inner integral is $-2 + 2x + 2x^2 - 2x^3$.



As a type II region, the region has to be split $\int_0^1 \int_0^y (x^2 - y)\ dxdy + \int_1^2 \int_0^{2-x} (x^2 - y)\ dxdy = -1/4 - 7/12 = -5/6$.

5) (4 points) Compute the probability that a quantum particle with energy $(k^2 + n^2)\hbar^2/(2m) = 5\hbar^2/(2m)$ is in the region $R = [0,\pi/2] \times [0,\pi/2] = \{(x,y) \mid 0 \leq x \leq \pi/2, 0 \leq y \leq \pi/2\}$ of the square box $[0,\pi] \times [0,\pi]$.

Check. The result will involve integrals

$$A = \int_0^\pi \int_0^\pi \sin^2(kx)\sin^2(ny)\ dxdy$$

as well as

$$B = \int_0^{\pi/2} \int_0^{\pi/2} \sin^2(kx)\sin^2(ny)\ dxdy$$

In the notes, you find an analogues computation for a different region R and a different energy level. Identities like $1 - 2\sin^2(x) = \cos(2x)$ are useful here.

Solution:
The integral $A = \pi^2/4$ is the integral of the wave function over the whole box, the integral $B = \pi^2/16$ is the integral over the region R. The result is $B/A = 1/4$. This result makes sense due to the symmetry of the wave.

Remarks

(You don't need to read these remarks to do the problems.)

A quantum mechanical particle confined to a region X in two dimensions is represented by a function $f(t,x,y)$ satisfying $\int\int_X f^2(t,x,y)\, dxdy = 1$. For each time t, the probability that the particle is in some subregion R is the double integral

$$\int\int_R f^2(t,x,y) \;\; dxdy \; .$$

According to the classical interpretation of quantum mechanics, particles don't have determined positions any more, the probabilities are all an experimenter can measure. If a particle is exposed to an external field $F(x,y) = \nabla V(x,y)$, then the evolution of the particle is given by the equation

$$\frac{i\hbar}{m} f_t(t,x,y) = f_{xx}(x,y) + f_{yy}(x,y) + V(x,y)f(x,y)$$

which is called the **Schrödinger equation**. A special case is if there is no force $F = \nabla V$. In that case, one can assume that $V(x,y) = 0$. Important are solutions $f(x,y)$ which satisfy the partial differential equation

$$f_{xx}(x,y) + f_{yy}(x,y) + V(x,y)f(x,y) = Ef(x,y) \; ,$$

where E is a number called the **energy**. In this case, the evolution of the particle is $\frac{i\hbar}{m} f_t(t,x,y) = Ef(t,x,y)$ which has a solution $e^{i\hbar t/m} f(0,x,y)$. It is a mathematical fact that for a bounded region X, not all energies E are allowed. They come in discrete steps, energies appear **"quantized"**. Quantized energies also appear for potentials like $V(x,y) = x^2 + y^2$, which is called the quantum mechanical oscillator or $V(x,y) = \frac{1}{\sqrt{x^2+y^2}}$ which is called the Coulomb potential. All what has been said works also in three dimensions. It is just that f depends now on three space variables and the double integrals will be replaced by triple integrals. If $V(x,y,z) = 1/|(x,y,z)|$, then the mathematics of the solutions to $f_{xx}(x,y,z) + f_{yy}(x,y,z) + f_{zz}(x,y,z) + V(x,y,z)f(x,y) = Ef(x,y)$ is the story of the **hydrogen atom**. The possible energy levels explains to a great deal the build-up of the periodic elements and so the stuff we are made of.

Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

1) Let M be a polygon in the plane where each edge is at a lattice point. Verify that the area A of the polygon satisfies $A = I + B/2 - 1$, where I is the number of lattice points inside the polygon and B is the number of lattice points at the boundary.

Solution:

2) The integral $\int_0^1 \arccos(\sqrt{x})\, dx$ can be written as a double integral $\int_0^1 \int_0^{\arccos(\sqrt{x})} dydx$. Calculate this integral.