This is part 1 (of 2) of the homework for the third week. It is due July 19 at the beginning of class.

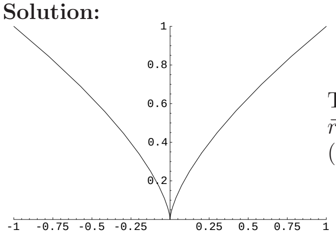
SUMMARY.

- $\vec{r}(t) = (x(t), y(t), z(t))$, $t \in [a, b]$ **curve** in space.
- $t \mapsto \vec{r}(t)$ is called **parametric representation** of the curve.
- $\vec{r}\,'(t) = (x'(t), y'(t), z'(t))$ **velocity**, $\vec{T}(t) = \vec{r}\,'(t)/|\vec{r}\,'(t)|$ **unit tangent vector**.
- $\vec{r}\,''(t) = (x''(t), y''(t), z''(t))$ **acceleration**.
- $\vec{R}(t) = \int_a^t \vec{r}(s)\ ds = (\int_a^t x(s)\ ds, \int_a^t y(s)\ ds, \int_a^t z(s)\ ds) + (C_1, C_2, C_3)$ **anti-derivative**, satisfies $\vec{R}\,'(t) = \vec{r}(t)$. (C_1, C_2, C_3) are arbitrary constants.
- $f(t)$ function, $\vec{r}(t)$ curve, $\frac{d}{dt} r(f(t)) = r'(f(t))f'(t)$. **chain rule**.

Homework Problems

1) (4 points)
Sketch the plane curve $\vec{r}(t) = (x(t), y(t)) = (t^3, t^2)$ for $t \in [-1, 1]$ by plotting the points for different values of t. Calculate its velocity $\vec{r}\,'(t)$ as well as its acceleration $\vec{r}\,''(t)$ at the point $t = 2$.

Solution:



The velocity is $\vec{r}\,'(t) = (3t^2, 2t)$. The acceleration is $\vec{r}\,''(t) = (6t, 2)$. At the time $t = 2$ we have $\vec{r}\,'(2) = (12, 4)$ and $\vec{r}\,''(2) = (12, 2)$.

2) (4 points) A device in a car measures the acceleration $\vec{r}\,''(t) = (\cos(t), -\cos(3t))$ at time t. Assume that the car is at the origin at time $t = 0$ and has zero speed at $t = 0$, what is its position $\vec{r}(t)$ at time t?

Solution:
$\vec{r}\,'(t) = (\sin(t), -\sin(3t)/3) + (C_1, C_2)$. Because the car has zero speed at time $t = 0$, we have $C_1 = C_2 = 0$.
From $\vec{r}\,'(t) = (\sin(t), -\sin(3t)/3)$, we obtain $\vec{r}(t) = (-\cos(t), +\cos(3t)/9) + (C_1, C_2)$. Because $\vec{r}(0) = (0, 0)$, we have $C_1 = 1, C_2 = -1/9$. $\vec{r}(t) = (-\cos(t) + 1, -\cos(3t)/9 - 1/9)$.

3) (4 points) Consider the curve $\vec{r}(t) = (x(t), y(t), z(t)) = (t^2, 1 + t, 1 + t^3)$.
a) (1) Verify that it passes through the point $(1, 0, 0)$.
b) (3) Find the velocity vector $\vec{r}\,'(t)$, the acceleration vector $\vec{r}\,''(t)$ as well as the jerk vector $\vec{r}\,'''(t)$ at the point $(1, 0, 0)$.

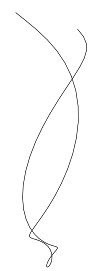
Solution:
a) Take $t = -1$.
b) $\vec{r}\,'(t) = (2t, 1, 3t^2)$,
$\vec{r}\,''(t) = (2, 0, 6t)$,
$\vec{r}\,'''(t) = (0, 0, 6)$
At time $t = -1$, we obtain $\vec{r}\,'(-1) = (-2, 1, 3)$,
$\vec{r}\,''(-1) = (2, 0, -6)$,
$\vec{r}\,'''(-1) = (0, 0, 6)$

4) (4 points) a) (2) Verify that the curve $\vec{r}(t) = (t\cos(t), 2t\sin(t), t^2)$ lies on the elliptic paraboloid $z = x^2 + y^2/4$.

b) (2) Use this fact to sketch the curve.

Solution:
a) Just plug in $x(t)^2 + y(t)^2 = z$.

b)



5) (4 points) Find the parameterization $\vec{r}(t) = (x(t), y(t), z(t))$ of the curve obtained by intersecting the cylinder $x^2 + y^2 = 9$ with the surface $z = xy$.

a) Write down the formula for the velocity vector $\vec{r}\,'(t)$.
b) If $f(t) = t^2$. Find $\frac{d}{dt}\vec{r}(f(t))$ using the chain rule.

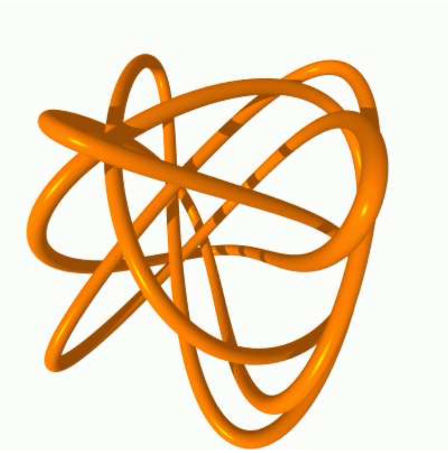
Solution:
We find first $x(t) = 3\cos(t), y(t) = 3\sin(t)$ using the first equation. Then get $z(t) = x(t)y(t) = 9\cos(t)\sin(t)$.
$\vec{r}(t) = (x(t), y(t), z(t)) = (3\cos(t), 3\sin(t), 9\cos(t)\sin(t)/2)$. The velocity vector is $\vec{r}'(t) = (x'(t), y'(t), z'(t)) = (-3sin(t), 3\cos(t), 9\cos^2(t) - 9\sin^2(t))$.
b) $d/dtr(f(t)) = r'(t^2)2t = (-3sin(t), 3\cos(t), 9\cos^2(t) - 9\sin^2(t))2t$.

Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

1) A closed curve in space is called a **knot**. Consider the space curve $\vec{r}(t) = (\sin(3t), \cos(4t), \cos(5t))$. Find the smallest interval $[a, b]$ such that this curve is a knot. Sketch the curve.

Hint. Try first without technology. If needed, peek at the website http : //www.math.harvard.edu/jdg or type ParametricPlot3D[{Sin[3t], Cos[4t], Cos[5t]}, {t, 0, 2Pi}] in Mathematica.



2) How could one verify that it is not possible to deform the knot $\vec{r}(t) = (\sin(3t), \cos(4t), \cos(5t))$ into the trivial knot $\vec{r}(t) = (\cos(t), \sin(t), 0)$ in such a way that during the deformation, the curve can never self-intersect?

Hint. Look at the possible types of closed curves which don't intersect the knot. How many different types are there for the trivial knot or for the given knot?