

This is part 3 (of 3) of the homework which is due July 5 at the beginning of class.

SUMMARY.

- $\vec{v} \times \vec{w} = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1)$ **cross product**.

- The following "matrix" helps to memorize compute the components:
$$\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}.$$

- $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}|\sin(\phi)$, where ϕ is the **angle** between vectors.

This is the area of parallelogram spanned by $\vec{v}$ and $\vec{w}$.

- $\vec{v} \times \vec{w}$ is **orthogonal** to $\vec{v}$ and to $\vec{w}$ with length $|\vec{v}||\vec{w}|\sin(\phi)$
- $\vec{u} \cdot (\vec{v} \times \vec{w})$ **triple scalar product**, signed volume of parallelepiped spanned by $\vec{u}, \vec{v}, \vec{w}$.
- $\vec{r}(t) = P + t\vec{v}$ **parametric equation** for a line, P a point, $\vec{v}$ is a vector.
- $\frac{(x-x_0)}{a} = \frac{(y-y_0)}{b} = \frac{(z-z_0)}{c}$ **symmetric equation** for a line.

Homework Problems

- (4 points)
 - (2) Find a the cross product $\vec{w}$ of $\vec{u} = (-3, -1, 2)$ and $\vec{v} = (-2, -2, 3)$.
 - (1) Find a unit vector $\vec{n}$ orthogonal to $\vec{u}$ and $\vec{v}$.
 - (1) Find the volume of the parallelepiped spanned by $\vec{u}, \vec{v}$ and $\vec{w}$.

Solution:

- $\vec{u} \times \vec{v} = \vec{w} = (1, 5, 4)$
- $|\vec{w}| = 42, \vec{n} = \vec{w}/|\vec{w}| = (1/\sqrt{42}, 5/\sqrt{42}, 4/\sqrt{42})$.
- $\vec{w} \cdot (\vec{u} \times \vec{v}) = \vec{w} \cdot \vec{w} = |\vec{w}|^2 = 42$.

- (4 points)
 - Verify the identity $|\vec{v} \times \vec{w}|^2 = |\vec{v}|^2|\vec{w}|^2 - (\vec{v} \cdot \vec{w})^2$ from the lecture.
 - Knowing $(\vec{v} \cdot \vec{w}) = |\vec{v}||\vec{w}|\cos(\phi)$, derive $|(\vec{v} \times \vec{w})| = |\vec{v}||\vec{w}|\sin(\phi)$

Solution:

- Direct, but a bit tedious computation.
- Use the identity $\cos^2(x) + \sin^2(x) = 1$.

- (4 points)
 - If $\vec{u} + \vec{v} + \vec{w} = \vec{0}$. Then $\vec{u} \times \vec{v} = \vec{v} \times \vec{w} = \vec{w} \times \vec{u}$.
 - Find $(\vec{u} + \vec{v}) \cdot (\vec{v} \times \vec{w})$ if $\vec{u}, \vec{v}, \vec{w}$ are unit vectors which are orthogonal to each other and $\vec{u} \times \vec{v} = \vec{w}$.
 - Assume you have a triangle in the plane which has edge points having integer coordinates. Show that the area of the triangle is an integer or half of an integer.

Solution:

- Build a triangle with the three vectors u, v, w . Each of the terms in the identity is twice the area of the triangle.
- The result is 1.
- The formula for the area is $|(A - C) \times (B - C)|/2$.

- (4 points) Given three vectors $\vec{u}, \vec{v}, \vec{w}$ with $V = \vec{u} \cdot (\vec{v} \times \vec{w}) \neq 0$. Define three new vectors

$$\begin{aligned} \vec{a} &= (\vec{v} \times \vec{w})/V \\ \vec{b} &= (\vec{w} \times \vec{u})/V \\ \vec{c} &= (\vec{u} \times \vec{v})/V. \end{aligned}$$

Verify that $\vec{a} \cdot (\vec{b} \times \vec{c}) = 1/V$.

Hint. You can use the identity $\vec{b} \times (\vec{u} \times \vec{v}) = (\vec{b} \cdot \vec{v})\vec{u} - (\vec{b} \cdot \vec{u})\vec{v}$ which holds in general. (If you have time, derive this identity (see challenge problems), but this is not required to get full credit for this problem).

Solution:

Focus first on $(\vec{b} \times \vec{c}) = \vec{b} \times (\vec{u} \times \vec{v})/V$ and use the hint to get $(\vec{b} \cdot \vec{v})\vec{u}/V - (\vec{b} \cdot \vec{u})\vec{v}/V$. The problem asks for the dot product of this with $\vec{a}$. Now, since $\vec{a}$ is orthogonal to the second term, we obtain $\vec{a}(\vec{b} \cdot \vec{v}) \cdot \vec{u}/V$. When plugging in the definitions of $\vec{a}$ and $\vec{b}$, we are left with $V^2/V^3 = 1/V$.

- (4 points)
 - (2) Find the parametric equation for the line which passes through the points $P = (1, 2, 3)$ and $Q = (3, 4, 5)$.
 - (2) Find the symmetric equation for the same line.

Solution:

- The vector $\vec{v} = (2, 2, 2)$ connects the two points. The parametric equation is $P + t\vec{v} = (1, 2, 3) + t(2, 2, 2) = (1 + 2t, 2 + 2t, 3 + 2t)$.
- $(x - 1)/2 = (y - 2)/2 = (z - 3)/2$.

Remarks

(You don't need to read these remarks to do the problems.)

To problem 4): three vectors whose triple scalar product does not vanish are called **non-coplanar**. Adding integer multiples of such vectors form a **lattice**. The points of the lattice are all points $n\vec{u} + m\vec{v} + k\vec{w}$, where n, m, k are integers.

The three new vectors $\vec{a}, \vec{b}$ and $\vec{c}$ defined in problem 5) define a new lattice which is called the **reciprocal lattice**. Crystallographers also denote them by $\vec{u}^*, \vec{v}^*$ and $\vec{w}^*$. What you have shown in 5) is that the volume V^* of the unit cells of the reciprocal lattice is the inverse $1/V$ of the volume V of the unit cell of the lattice itself. The reciprocal lattice is essential for the study of crystal lattices and their diffraction properties which can be measured by shooting X-rays onto them. A convenient way to link the structure of the material to its diffraction pattern is through the reciprocal lattice.

Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

- 1) Prove the following identity for vectors $\vec{a}, \vec{b}, \vec{c}$ in space:

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

- 2) Find a general formula for the volume of a tetrahedron with corners P, Q, R, S .

Hint. Find first a formula for the area of one of its triangular faces, and then a formula for the distance from the fourth point to that face.

The change of the angular momentum $\vec{L}$ in the torque satisfies

3)
$$\frac{d}{dt}\vec{L} = \vec{L} \times \vec{\Omega} ,$$

where $\vec{\Omega}$ is the angular velocity vector. Verify that the length of $\vec{L}$ does not change in time.

