SPHERICAL INTEGRALS
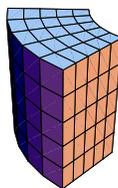
REMINDER: POLAR AND CYLINDRICAL COORDINATES.

$$\int \int_R f(r,\theta) \,\boxed{r}\, d\theta dr \ .$$

Cylindrical coordinates are obtained by taking polar coordinates in the x-y plane and leave the z coordinate. With $T(r,\theta,z) = (r\cos(\theta), r\sin(\theta), z)$. the integration factor r is the same as in polar coordinates.
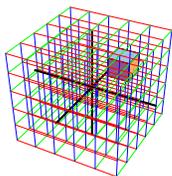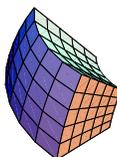
$$\int \int \int_{T(R)} f(x,y,z)\, dxdydz = \int \int \int_R g(r,\theta,z)\,\boxed{r}\, drd\theta dz$$

SPHERICAL COORDINATES.
Spherical coordinates use ρ, the distance to the origin as well as two angles: θ the polar angle and ϕ, the angle between the vector and the z axis. The coordinate change is
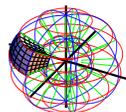
$$T : (x,y,z) = (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi)) \ .$$

The integration factor can be seen by measuring the volume of a **spherical wedge** which is $d\rho, \rho\sin(\phi)\,d\theta, \rho d\phi = \rho^2\sin(\phi)d\theta d\phi d\rho$.

$$\int \int \int_{T(R)} f(x,y,z)\, dxdydz = \int \int \int_R g(\rho,\theta,z)\,\boxed{\rho^2\sin(\phi)}\, d\rho d\theta d\phi$$
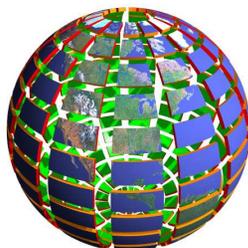
$$T : (\rho,\theta,\phi)$$
$$\mapsto$$
$$(\rho\cos(\theta)\sin(\phi),$$
$$\rho\sin(\theta)\sin(\phi),$$
$$\rho\cos(\phi))$$

VOLUME OF SPHERE.
A sphere of radius R has the volume

$$\int_0^R \int_0^{2\pi} \int_0^\pi \rho^2\sin(\phi)\, d\phi d\theta d\rho \ .$$

The most inner integral $\int_0^\pi \rho^2\sin(\phi)d\phi = -\rho^2\cos(\phi)|_0^\pi = 2\rho^2$.
The next layer is, because ϕ does not appear: $\int_0^{2\pi} 2\rho^2\, d\phi = 4\pi\rho^2$.
The final integral is $\int_0^R 4\pi\rho^2\, d\rho = 4\pi R^3/3$.

MOMENT OF INERTIA.
The moment of inertia of a body G with respect to an axis L is defined as the triple integral $\int \int \int_G r(x,y,z)^2\, dzdydx$, where $r(x,y,z) = R\sin(\phi)$ is the distance from the axes L. For a sphere of radius R we obtain with respect to the z-axis:

$$I = \int_0^R \int_0^{2\pi} \int_0^\pi \rho^2\sin^2(\phi)\rho^2\sin(\phi)\, d\phi d\theta d\rho$$

$$= (\frac{1}{3}\sin^3(\phi)_0^\pi)(\int_0^R \rho^4\, dr)(\int_0^{2pi} d\theta)$$

$$= \frac{4}{3} \cdot \frac{R^5}{5} \cdot 2\pi = \frac{8\pi R^5}{15} = \frac{VR^2}{5}$$

If the sphere rotates with angular velocity ω, then $I\omega^2/2$ is the **kinetic energy** of that sphere. Example: the moment of inertia of the earth is $810^{37}kgm^2$. The angular velocity is $\omega = 1/day = 1/(86400s)$. The rotational energy is $810^{37}kgm^2/(7464960000s^2) \sim 10^{28}J = 10^{25}kJ \sim 2.510^{24}kcal$.

How long would you have to run on a treadmill to accumulate this energy if you could make 2'500 kcal/hour? We would have to run 10^{21} hours $= 3.610^{24}$ seconds. Note that the universe is about 10^{17} seconds old. If all the 6 million people in Massachusetts would have run since the big bang on a treadmill, they could have produced the necessary energy to bring the earth to the current rotation. To make classes pass faster, we need to spin the earth more and just to add some more treadmills ... To the right you see a proposal for the science center.
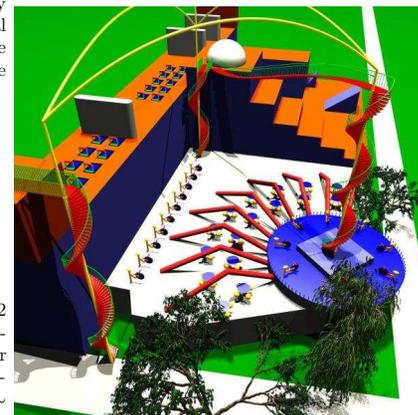
DIAMOND.
Find the volume and the center of mass of a diamond, the intersection of the unit sphere with the cone given in cylindrical coordinates as $z = \sqrt{3}r$.
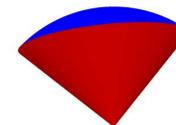
Solution: we use spherical coordinates to find the center of mass $(\overline{x}, \overline{y}, \overline{z})$:

$V = \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^2\sin(\phi)\, d\phi d\theta d\rho = \frac{(1-\frac{\sqrt{3}}{2})}{3}2\pi$

$\overline{x} = \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3\sin^2(\phi)\cos(\theta)\, d\phi d\theta d\rho/V = 0$

$\overline{y} = \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3\sin^2(\phi)\sin(\theta)\, d\phi d\theta d\rho/V = 0$

$\overline{z} = \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3\cos(\phi)\sin(\phi)\, d\phi d\theta d\rho/V = \frac{2\pi}{32V}$

PROBLEM
Find $\int \int \int_R z^2\, dV$ for the solid obtained by intersecting $\{1 \le x^2+y^2+z^2 \le 4\}$ with the double cone $\{z^2 \ge x^2+y^2\}$.

Solution: since the result for the double cone is twice the result for the single cone, we work with the diamond shaped region R in $\{z > 0\}$ and multiply the result at the end with 2.
In spherical coordinates, the solid R is given by $1 \le \rho \le 2$ and $0 \le \phi \le \pi/4$. With $z = \rho\cos(\phi)$, we have

$$\int_1^2 \int_0^{2\pi} \int_0^{\pi/4} \rho^4\cos^2(\phi)\sin(\phi)\, d\phi d\theta d\rho = (\frac{2^5}{5} - \frac{1^5}{5})2\pi(\frac{-\cos^3(\phi)}{3}|_0^{\pi/}$$

The result for the double cone is $\boxed{4\pi(31/5)(1 - 1/\sqrt{2}^3)}$.