

REVIEW BEYOND VECTOR FIELDS		Maths21a,O.Knill
INTEGRATION. Line integral: $\int_C F \cdot ds = \int_a^b F(r(t)) \cdot r'(t) dt$ Surface integral $\int \int_S f dS = \int_a^b \int_c^d f(r(u,v)) r_u(u,v) \times r_v(u,v) dudv$ Flux integral: $\int \int_S F \cdot dS = \int_a^b \int_c^d F(r(u,v)) \cdot r_u(u,v) \times r_v(u,v) dudv$ Double integral: $\int \int_R f dA = \int_a^b \int_c^d f(x,y) dxdy.$ Triple integral: $\int \int \int_R f dV = \int_a^b \int_c^d \int_o^p f(x,y,z) dxdydz.$	Area $\int \int_R 1 dA = \int \int_R 1 dxdy$ Length $\int_a^b r'(t) dt$ Surface area $\int \int 1 dS = \int \int r_u \times r_v dudv$ Volume $\int \int \int_B 1 dV = \int \int \int_B 1 dxdydz$	
DIFFERENTIATION. Derivative: $f'(t) = \dot{f}(t) = d/dt f(t).$ Partial derivative: $f_x(x,y,z) = \frac{\partial f}{\partial x}(x,y,z).$ Gradient: $\text{grad}(f) = (f_x, f_y, f_z)$ Curl in 2D: $\text{curl}(F) = \text{curl}((M,N)) = N_x - M_y$ Curl in 3D: $\text{curl}(F) = \text{curl}(M,N,P) = (P_y - N_z, M_z - P_x, N_x - M_y)$ Div: $\text{div}(F) = \text{div}(M,N,P) = M_x + N_y + P_z.$	IDENTITIES. $\text{div}(\text{curl}(F)) = 0$ $\text{curl}(\text{grad}(f)) = (0,0,0)$ $\text{div}(\text{grad}(f)) = \Delta f.$	
NOTIONSS OF CONSERVATIVE. 1) Gradient: $F = \text{grad}(f).$ 2) Closed curve property: $\int_C F \cdot dr = 0$ for any closed curve. 3) Conservative: C_i paths from A to B, then $\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr.$ 4) Mixed derivative property: $\text{curl}(F) = 0$ in simply connected regions.	<pre> graph TD A[Gradient field] --> B[Conservative] B --> C[Closed loop property] C --> D[Passes component test] D --> A B <--> C </pre>	
TOPOLOGY. 1) Interior of region D: points which have a neighborhood contained in D. 2) Boundary of a curve: endpoints of curve, boundary of 2D region D: curves which bound the region, boundary of a solid D: surfaces which bound the solid. 3) Simply connected region D: a closed curve in D can be deformed within the interior of D to a point. 4) Closed curve Curve without boundary. 5) Closed surface surface without boundary.		
LINE INTEGRAL THEOREM. If $C : r(t) = (x(t), y(t), z(t)), t \in [a, b]$ is a curve and f is a function either in 3D or the plane. Then	$\int_C \nabla f \cdot ds = f(r(b)) - f(r(a))$	
CONSEQUENCES. 1) For closed curves the line integral $\int_C \nabla f \cdot ds$ is zero. 2) Gradient fields are conservative: if $F = \nabla f$, then the line integral between two points P and Q is path independent. 3) The theorem holds in any dimension. In one dimension, it reduces to the fundamental theorem of calculus $\int_a^b f'(x) dx = f(b) - f(a)$ 4) The theorem justifies the name conservative for gradient vector fields. 5) The term "potential" was coined by George Green (1783-1841).		
PROBLEM. Let $f(x,y,z) = x^2 + y^4 + z$. Find the line integral of the vector field $F(x,y,z) = \nabla f(x,y,z)$ along the path $r(t) = (\cos(5t), \sin(2t), t^2)$ from $t = 0$ to $t = 2\pi$.		
SOLUTION. $r(0) = (1, 0, 0)$ and $r(2\pi) = (1, 0, 4\pi^2)$ and $f(r(0)) = 1$ and $f(r(2\pi)) = 1 + 4\pi^2$. FTLI gives $\int_C \nabla f ds = f(r(2\pi)) - f(r(0)) = 4\pi^2$.		
GREEN'S THEOREM. If R is a region with boundary C and $F = (M, N)$ is a vector field, then	$\int \int_R \text{curl}(F) dA = \int_C F \cdot ds$	
REMARKS. 1) Useful to swap 2D integrals to 1D integrals or the other way round. 2) The curve is oriented in such a way that the region is to your left. 3) The region has to have piecewise smooth boundaries (i.e. it should not look like the Mandelbrot set). 4) If $C : t \mapsto r(t) = (x(t), y(t))$, the line integral is $\int_a^b (M(x(t), y(t)), N(x(t), y(t))) \cdot (x'(t), y'(t)) dt$. 5) Green's theorem was found by George Green (1793-1841) in 1827 and by Mikhail Ostrogradski (1801-1862). 6) If $\text{curl}(F) = 0$ in a simply connected region, then the line integral along a closed curve is zero. If two curves connect two points then the line integral along those curves agrees. 7) Taking $F(x,y) = (-y, 0)$ or $F(x,y) = (0, x)$ gives area formulas.		
PROBLEM. Find the line integral of the vector field $F(x,y) = (x^4 + \sin(x) + y, x + y^3)$ along the path $r(t) = (\cos(t), 5 \sin(t) + \log(1 + \sin(t)))$, where t runs from $t = 0$ to $t = \pi$.		
SOLUTION. $\text{curl}(F) = 0$ implies that the line integral depends only on the end points $(0, 1), (0, -1)$ of the path. Take the simpler path $r(t) = (-t, 0), t = [-1, 1]$, which has velocity $r'(t) = (-1, 0)$. The line integral is $\int_{-1}^1 (t^4 - \sin(t), -t) \cdot (-1, 0) dt = -t^5/5 _{-1}^1 = -2/5$.		
REMARK. We could also find a potential $f(x,y) = x^5/5 - \cos(x) + xy + y^5/4$. It has the property that $\text{grad}(f) = F$. Again, we get $f(0, -1) - f(0, 1) = -1/5 - 1/5 = -2/5$.		
STOKES THEOREM. If S is a surface in space with boundary C and F is a vector field, then	$\int \int_S \text{curl}(F) \cdot dS = \int_C F \cdot ds$	
REMARKS. 1) Stokes theorem implies Greens theorem if F is z independent and S is contained in the z-plane. 2) The orientation of C is such that if you walk along C and have your head in the direction, where the normal vector $r_u \times r_v$ of S, then the surface is to your left. 3) Stokes theorem was found by André Ampère (1775-1836) in 1825 and rediscovered by George Stokes (1819-1903). 4) The flux of the curl of a vector field does not depend on the surface S, only on the boundary of S. This is analogue to the fact that the line integral of a gradient field only depends on the end points of the curve. 5) The flux of the curl through a closed surface like the sphere is zero: the boundary of such a surface is empty.		
PROBLEM. Compute the line integral of $F(x,y,z) = (x^3 + xy, y, z)$ along the polygonal path C connecting the points $(0, 0, 0), (2, 0, 0), (2, 1, 0), (0, 1, 0)$.		
SOLUTION. The path C bounds a surface $S : r(u,v) = (u, v, 0)$ parameterized by $R = [0, 2] \times [0, 1]$. By Stokes theorem, the line integral is equal to the flux of $\text{curl}(F)(x,y,z) = (0, 0, -x)$ through S. The normal vector of S is $r_u \times r_v = (1, 0, 0) \times (0, 1, 0) = (0, 0, 1)$ so that $\int \int_S \text{curl}(F) dS = \int_0^2 \int_0^1 (0, 0, -u) \cdot (0, 0, 1) dudv = \int_0^2 \int_0^1 -u dudv = -2$.		
GAUSS THEOREM. If S is the boundary of a region B in space with boundary S and F is a vector field, then	$\int \int \int_B \text{div}(F) dV = \int \int_S F \cdot dS$	
REMARKS. 1) Gauss theorem is also called divergence theorem. 2) Gauss theorem can be helpful to determine the flux of vector fields through surfaces. 3) Gauss theorem was discovered in 1764 by Joseph Louis Lagrange (1736-1813), later it was rediscovered by Carl Friedrich Gauss (1777-1855) and by George Green. 4) For divergence free vector fields F, the flux through a closed surface is zero. Such fields F are also called incompressible or source free.		
PROBLEM. Compute the flux of the vector field $F(x,y,z) = (-x, y, z^2)$ through the boundary S of the rectangular box $[0, 3] \times [-1, 2] \times [1, 2]$.		
SOLUTION. By Gauss theorem, the flux is equal to the triple integral of $\text{div}(F) = 2z$ over the box: $\int_0^3 \int_{-1}^2 \int_1^2 2z dxdydz = (3-0)(2-(-1))(4-1) = 27$.		