GLOSSARY CHECKLIST — **Maths21a, Oliver Knill**

Geometry of Space

- [ ] coordinates and vectors in the plane and in space
- [ ] $v = (v_1, v_2, v_3), w = (w_1, w_2, w_3)$, $v + w = (v_1 + w_1, v_2 + w_2, v_3 + w_3)$
- [ ] dot product $v.w = v_1w_1 + v_2w_2 + v_3w_3 = |v||w|\cos(\alpha)$
- [ ] cross product, $v.(v \times w) = 0$, $w.(v \times w) = 0$, $|v \times w| = |v||w|\sin(\alpha)$
- [ ] triple scalar product $u \cdot (v \times w)$ volume of parallelepiped
- [ ] parallel vectors $v \times w = 0$, orthogonal vectors $v \cdot w = 0$
- [ ] scalar projection $\text{comp}_w(v) = v \cdot w/|w|$
- [ ] vector projection $\text{proj}_w(v) = (v \cdot w)w/|w|^2$
- [ ] completion of square: example $x^2 - 4x + y^2 = 1$ is equivalent to $(x-2)^2 + y^2 = -3$
- [ ] distance $d(P, Q) = |\vec{PQ}| = \sqrt{(P_1 - Q_1)^2 + (P_2 - Q_2)^2 + (P_3 - Q_3)^2}$

Lines, Planes, Functions

- [ ] symmetric equation of line $\frac{(x-x_0)}{a} = \frac{(y-y_0)}{b} = \frac{z-z_0}{c}$
- [ ] plane $ax + by + cz = d$
- [ ] parametric equation for line $\vec{x} = \vec{x}_0 + t\vec{v}$
- [ ] parametric equation for plane $\vec{x} = \vec{x}_0 + t\vec{v} + s\vec{w}$
- [ ] switch from parametric to implicit descriptions for lines and planes
- [ ] domain and range of functions $f(x, y)$
- [ ] graph $G = \{(x, y, f(x, y))\}$
- [ ] intercepts: intersections of G with coordinate axes
- [ ] traces: intersections with coordinate planes
- [ ] generalized traces: intersections with $\{x = c\}, \{y = c\}$ or $\{z = c\}$
- [ ] quadrics: ellipsoid, paraboloid, hyperboloids, cylinder, cone, hyperboloid paraboloid
- [ ] plane $ax + by + cz = d$ has normal $\vec{n} = (a, b, c)$
- [ ] line $\frac{(x-x_0)}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$ contains $\vec{v} = (a, b, c)$
- [ ] sets $g(x, y, z) = c$ describe surfaces, example graphs $g(x, y, z) = z - f(x, y)$
- [ ] linear equation like $2x + 3y + 5z = 7$ defines plane
- [ ] quadratic equation like $x^2 - 2y^2 + 3z^2 = 4$ defines quadric surface
- [ ] distance point-plane: $d(P, \Sigma) = |(\vec{PQ}) \cdot \vec{n}|/|\vec{n}|$
- [ ] distance point-line: $d(P, L) = |(\vec{PQ}) \times \vec{u}|/|\vec{u}|$
- [ ] distance line-line: $d(L, M) = |(\vec{PQ}) \cdot (\vec{u} \times \vec{v})|/|\vec{u} \times \vec{v}|$
- [ ] finding plane through three points A, B, C: find normal vector $(a, b, c) = \vec{AB} \times \vec{CB}$

Curves

- [ ] plane and space curves $\vec{r}(t)$
- [ ] velocity $\vec{r}'(t)$, acceleration $\vec{r}''(t)$
- [ ] unit tangent vector $\vec{T}(t) = \vec{r}'(t)/|\vec{r}'(t)|$
- [ ] $\vec{r}'(t)$ is tangent to the curve
- [ ] $\vec{v} = \vec{r}'$ then $\vec{r} = \int_0^t \vec{v}\, dt + \vec{c}$
- [ ] $\vec{r}(t) = (f(t)\cos(t), r(t)\sin(t))$ polar curve to polar graph $r = f(\theta)$.
- [ ] $\kappa(t) = |\vec{T}'(t)|/|\vec{r}'(t)|$ curvature

Surfaces

- [ ] polar coordinates $(x, y) = (r\cos(\theta), r\sin(\theta))$
- [ ] cylindrical coordinates $(x, y, z) = (r\cos(\theta), r\sin(\theta), z)$
- [ ] spherical coordinates $(x, y, z) = (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi))$
- [ ] $g(r, \theta) = 0$ polar curve, especially $r = f(\theta)$, polar graphs
- [ ] $g(r, \theta, z) = 0$ cylindrical surface, i.e. $r = f(z, \theta)$ or $r = f(z)$ surface of revolution
- [ ] $g(\rho, \theta, \phi) = 0$ spherical surface especially $\rho = f(\theta, \phi)$
- [ ] $f(x, y) = c$ level curves of $f(x, y)$, normal vectors are $\nabla f(x, y)$
- [ ] $g(x, y, z) = c$ level surfaces of $g(x, y, z)$, normal vectors are $\nabla f(x, y, z)$
- [ ] circle: $x^2 + y^2 = r^2$, $\vec{r}(t) = (r\cos t, r\sin t)$.
- [ ] ellipse: $x^2/a^2 + y^2/b^2 = 1$, $\vec{r}(t) = (a\cos t, b\sin t)$
- [ ] sphere: $x^2 + y^2 + z^2 = r^2$, $\vec{r}(u, v) = (r\cos u \sin v, r\sin u \sin v, r\cos v)$
- [ ] ellipsoid: $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, $\vec{r}(u, v) = (a\cos u \sin v, b\sin u \sin v, c\cos v)$
- [ ] line: $ax + by = d$, $\vec{r}(t) = (t, d/b - ta/b)$
- [ ] plane: $ax + by + cz = d$, $\vec{r}(u, v) = \vec{r}_0 + u\vec{v} + v\vec{w}, (a, b, c) = \vec{v} \times \vec{w}$
- [ ] surface of revolution: $r(\theta, z) = f(z)$, $\vec{r}(u, v) = (f(v)\cos(u), f(v)\sin(u), v)$
- [ ] graph: $g(x, y, z) = z - f(x, y) = 0$, $\vec{r}(u, v) = (u, v, f(u, v))$

Partial Derivatives

- [ ] $f_x(x, y) = \frac{\partial}{\partial x} f(x, y)$ partial derivative
- [ ] partial differential equation PDE: $F(f, f_x, f_t, f_{xx}, f_{tt}) = 0$
- [ ] $f_t = f_{xx}$ heat equation
- [ ] $f_{tt} - f_{xx} = 0$ wave equation
- [ ] $f_x - f_t = 0$ transport equation
- [ ] $f_{xx} + f_{yy} = 0$ Laplace equation
- [ ] $L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ linear approximation
- [ ] tangent line: $L(x, y) = L(x_0, y_0)$, $ax + by = d$ with $a = f_x(x_0, y_0), b = f_y(x_0, y_0)$, $d = ax_0 + by_0$
- [ ] tangent plane: $L(x, y, z) = L(x_0, y_0, z_0)$
- [ ] estimate $f(x, y, z)$ by $L(x, y, z)$ near (x_0, y_0, z_0)
- [ ] $|f(x, y) - L(x, y)|$ in box R around (x_0, y_0) is $\leq M(|x - x_0| + |y - y_0|)^2/2$, where M is the maximal value of $|f_{xx}(x, y)|, |f_{xy}(x, y)|, |f_{yy}(x, y)|$ in R.
- [ ] $f(x, y)$ called differentiable if f_x, f_y are continuous
- [ ] $f_{xy} = f_{yx}$ Clairot's theorem
- [ ] $\vec{r}_u(u, v)$, $\vec{r}_v$ tangent to surface $\vec{r}(u, v)$

Gradient

- [ ] $\nabla f(x, y) = (f_x, f_y)$, $\nabla f(x, y, z) = (f_x, f_y, f_z)$, gradient
- [ ] $D_v f = \nabla i f \cdot v$ directional derivative
- [ ] $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$ chain rule
- [ ] $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface $f(x, y, z) = c$ which contains (x_0, y_0, z_0).
- [ ] $\frac{d}{dt} f(\vec{x} + t\vec{v}) = D_v f$ by chain rule
- [ ] $\frac{x-x_0}{f_x(x_0,y_0,z_0)} = \frac{y-y_0}{f_y(x_0,y_0,z_0)} = \frac{z-z_0}{f_z(x_0,y_0,z_0)}$ normal line to surface $f(x, y, z) = c$ at (x_0, y_0, z_0)
- [ ] $(x - x_0)f_x(x_0, y_0, z_0) + (y - y_0)f_y(x_0, y_0, z_0) + (z - z_0)f_z(x_0, y_0, z_0) = 0$ tangent plane at (x_0, y_0, z_0)
- [ ] directional derivative is maximal in the $\vec{v} = \nabla f$ direction
- [ ] $f(x, y)$ increases, if we walk on the xy-plane in the ∇f direction
- [ ] partial derivatives are special directional derivatives
- [ ] if $D_v f(\vec{x}) = 0$ for all $\vec{v}$, then $\nabla f(\vec{x}) = \vec{0}$

Extrema

- $\nabla f(x,y) = (0,0)$, critical point or stationary point
- $D = f_{xx} f_{yy} - f_{xy}^2$ discriminant or Hessian determinant
- $f(x_0,y_0) \geq f(x,y)$ in a neighborhood of (x_0,y_0) local maximum
- $f(x_0,y_0) \leq f(x,y)$ in a neighborhood of (x_0,y_0) local minimum
- $\nabla f(x,y) = \lambda \nabla g(x,y), g(x,y) = c$, λ Lagrange equations
- $\nabla f(x,y,z) = \lambda \nabla g(x,y,z), g(x,y,z) = c$, λ Lagrange equations
- second derivative test: $\nabla f = (0,0), D > 0, f_{xx} < 0$ **local max**, $\nabla f = (0,0), D > 0, f_{xx} > 0$ **local min**, $\nabla f = (0,0), D < 0$ **saddle**

Double Integrals

- $\int \int_R f(x,y)\, dA$ double integral
- $\int_a^b \int_c^d f(x,y)\, dydx$ integral over rectangle
- $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dydx$ type I region
- $\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dxdy$ type II region
- $\int \int_R f(r,\theta) \boxed{r}\, dr d\theta$ polar coordinates
- $\int \int_R |\vec{r}_u \times \vec{r}_v|\, dudv$ surface area
- $\int_a^b \int_c^d f(x,y)\, dydx = \int_c^d \int_a^b f(x,y)\, dxdy$ Fubini
- $\int \int_R 1\, dxdy$ area of region R
- $\int \int_R f(x,y)\, dxdy$ volume of solid bounded by graph(f) xy-plane

Triple Integrals

- $\int \int \int_R f(x,y,z)\, dV$ triple integral
- $\int_a^b \int_c^d \int_u^v f(x,y,z)\, dydx$ integral over rectangular box
- $\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y)\, dzdydx$ type I region
- $f(r,\theta,z) \boxed{r}\; dzdrd\theta$ cylindrical coordinates
- $\int \int \int_R f(\rho,\theta,z) \boxed{\rho^2 \sin(\phi)}\; dzdrd\theta$ spherical coordinates
- $\int_a^b \int_c^d \int_u^v f(x,y,z)\, dzdydx = \int_u^v \int_c^d \int_a^b f(x,y,z)\, dxdydz$ Fubini
- $V = \int \int \int_R 1\, dV$ volume of solid R
- $M = \int \int \int_R \rho(x,y,z)\, dV$ mass of solid R with density ρ
- $(\int \int \int_R x\, dV/V, \int \int \int_R y\, dV/V, \int \int \int_R z\, dV/V)$ center of mass

Line Integrals

- $F(x,y) = (P(x,y), Q(x,y))$ vector field in the plane
- $F(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ vector field in space
- $\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t)\, dt$ line integral
- Use also notation $\int_C F \cdot T ds$ and $\int P dx + Q dy + R dz$.
- $F(x,y) = \nabla f(x,y)$ gradient field = potential field = conservative

Fundamental theorem of line integrals

- FTL: $F(x,y) = \nabla f(x,y)$, $\int_a^b F(r(t)) \cdot r'(t)\, dt = f(r(b)) - f(r(a))$
- Closed loop property $\int_C F\, dr = 0$, for all closed curves C
- Always equivalent are: closed loop property, conservativeness and gradient field
- Mixed derivative test $\text{curl}(F) \neq 0$ assures F is not a gradient field.
- In simply connected domains, $\text{curl}(F) = 0$ implies conservativeness.

Green's Theorem

- $F(x,y) = (P,Q)$, curl in two dimensions: $\text{curl}(F) = Q_x - P_y = \nabla \times F$.
- Green's theorem: C boundary of R, then $\int_C F \cdot dr = \int \int_R \text{curl}(F)\, dxdy$
- Area computation: Take F with $\text{curl}(F) = N_x - M_y = 1$ like $F = (-y,0)$ or $F = (0,x)$ or $F = (-y,x)/2$.
- Greens theorem is useful to compute difficult line integrals or difficult 2D integrals.

Flux integrals

- $F(x,y,z)$ vector field, $S = r(R)$ parametrized surface
- $r_u \times r_v$ normal vector, $\vec{n} = \frac{r_u \times r_v}{|r_u \times r_v|}$ unit normal vector
- $r_u \times r_v\, dudv = \vec{dS} = \vec{n}\, dS$ normal surface element
- $\int \int_S F\, \vec{dS} = \int \int_S F(r(u,v)) \cdot (r_u \times r_v)\, dudv$ flux integral
- Use also notation $\int \int_S F \cdot n d\sigma$

Stokes Theorem

- $F(x,y,z) = (P,Q,R)$, $\text{curl}(P,Q,R) = (R_y - Q_z, P_z - R_x, Q_x - P_y) = \nabla \times F$
- Stokes's theorem: C boundary of surface S, then $\int_C F \cdot dr = \int \int_S \text{curl}(F) \cdot dS$
- Stokes theorem is useful to compute difficult flux integrals of curl(F) or difficult line integrals.

Div Grad Curl

- $\nabla = (\partial_x, \partial_y, \partial_z)$, $\text{grad}(F) = \nabla f$, $\text{curl}(F) = \nabla \times F$, $\text{div}(F) = \nabla \cdot F$
- $\text{div}(\text{curl}(F)) = 0$
- $\text{curl}(\text{grad}(F)) = \vec{0}$
- $\text{div}(\text{grad}(f)) = \Delta f$.

Divergence Theorem

- $\text{div}(P,Q,R) = P_x + Q_y + R_z = \nabla \cdot F$
- Divergence theorem: solid E, boundary S then $\int \int_S F \cdot dS = \int \int \int_E \text{div}(F)\, dV$
- The divergence theorem is useful to compute difficult flux integrals or difficult 3D integrals.

Some topology

- Simply connected region D: can deform any closed curve within D to a point on curve.
- Interior of a region D: points in D for which small neighborhood is still in D.
- Boundary of a curve: the end points of the curve if they exist.
- Boundary of a surface S are curves which bound the surface, points in the surface which correspond to parameters (u,v) which are not in the interior of the parametrization domain.
- Boundary of a solid D: the surfaces which bound the solid, points in the solid which are not in the interior of D.
- Closed surface: a surface without boundary like for example the sphere.
- Closed curve: a curve with no boundary like for example a knot.

Some surface parametrizations

- Sphere of radius ρ: $r(u,v) = (\rho \cos(u) \sin(v), \rho \sin(u) \sin(v), \rho \cos(v))$
- Graph of function $f(x,y)$: $f(u,v) = (u, v, f(u,v))$
- Graph of function $f(\phi, r)$ in polar: $f(u,v) = (v \cos(u), v \sin(u), f(u,v))$
- Plane containing P and vectors $\vec{u}, \vec{v}$: $r(u,v) = P + u\vec{u} + v\vec{v}$
- Surface of revolution: distance $g(z)$ of $z - axes : r(u,v) = (g(v) \cos(u), g(v) \sin(u), v)$
- Ex: Cylinder: $r(u,v) = (\cos(u), \sin(u), v)$.
- Ex: Cone: $r(u,v) = (v \cos(u), v \sin(u), v)$.
- Ex: Paraboloid: $r(u,v) = (\sqrt{r} \cos(u), \sqrt{r} \sin(u), v)$.