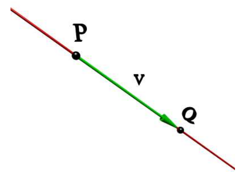
LINES and PLANES — **Maths21a, O. Knill**

LINES. A point P and a vector $\vec{v}$ define a line L. It is the set of points

$$\boxed{L = \{P + t\vec{v}, \text{ where } t \text{ is a real number }\}}$$



The line contains the point P and points into the direction $\vec{v}$.

EXAMPLE. $L = \{(x, y, z) = (1, 1, 2) + t(2, 4, 6)\ \}$.

This description is called the **parametric equation** for the line.

EQUATIONS OF LINE. We can write $(x, y, z) = (1, 1, 2) + t(2, 4, 6)$ so that $x = 1 + 2t, y = 1 + 4t, z = 2 + 6t$. If we solve the first equation for t and plug it into the other equations, we get $y = 1 + (2x - 2), z = 2 + 3(2x - 2)$. We can therefore describe the line also as

$$L = \{(x, y, z) \mid y = 2x - 1, z = 6x - 4\ \}$$

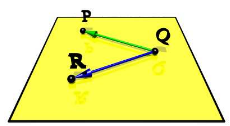
SYMMETRIC EQUATION. The line $\vec{r} = P + t\vec{v}$ with $P = (x_0, y_0, z_0)$ and $\vec{v} = (a, b, c)$ satisfies the **symmetric equations** $\boxed{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}}$ (every expression is equal to t).

PROBLEM. Find the equations for the line through the points $P = (0, 1, 1)$ and $Q = (2, 3, 4)$.

SOLUTION. The parametric equations are $(x, y, z) = (0, 1, 1) + t(2, 2, 3)$ or $x = 2t, y = 1 + 2t, z = 1 + 3t$. Solving each equation for t gives the symmetric equations $x/2 = (y - 1)/2 = (z - 1)/3$.

PLANES. A point P and two vectors $\vec{v}, \vec{w}$ define a plane Σ. It is the set of points

$$\boxed{\Sigma = \{P + t\vec{v} + s\vec{w}, \text{ where } t, s \text{ are real numbers }\}}$$



The line contains the point P.

EXAMPLE. $\Sigma = \{(x, y, z) \mid (1, 1, 2) + t(2, 4, 6) + s(1, 0, -1)\ \}$.

This is called the **parametric description** of a plane.

EQUATION OF PLANE. Given a plane as a parametric equation $P = Q + t\vec{v} + s\vec{w}$. The vector $\vec{n} = \vec{v} \times \vec{w}$ is orthogonal to both $\vec{n}$ and $\vec{w}$. Because also the vector $PQ = Q - P$ is perpendicular to $\vec{n}$ we have $(Q - P) \cdot \vec{n} = 0$. With $Q = (x_0, y_0, z_0)$, $P = (x, y, z)$, and $\vec{n} = (a, b, c)$, this means $ax + by + cz = ax_0 by_0 + cz_0 = d$. The plane is therefore described by a single equation $\boxed{ax + by + cz = d}$.

PROBLEM. Find the equation of a plane which contains the three points $P = (-1, -1, 1), Q = (0, 1, 1), R = (1, 1, 3)$.

SOLUTION. The plane contains the two vectors $\vec{v} = (1, 2, 0)$ and $\vec{w} = (2, 2, 2)$. We have $\vec{n} = (4, -2, -2)$ and the equation is $4x - 2y - 2z = d$. The constant d is obtained by plugging in one point: $\boxed{4x - 2y - 2z = -4}$.

LINES AND PLANES IN MATHEMATICA.
Plotting a line: `ParametricPlot3D[{1,1,1} + t{3,4,5}, {t, -2, 2}]`

Plotting a plane: `ParametricPlot3D[{1,1,1} + t{3,4,5} + s{1,2,3}, {t, -2, 2}, {s, -2, 2}]`

Finding the equation of a plane

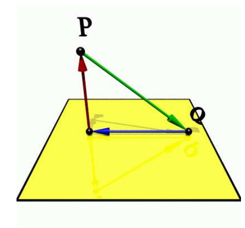
`P = {-1, -1, 1}; Q = {0, 1, 1}; R = {1, 1, 3}; n = Cross[Q - P, R - P]; n.{x, y, z} - n.P`

DISTANCE POINT-PLANE (3D). If P is a point in space and $\vec{n} \cdot \vec{x} = d$ is a plane containing a point Q, then

$$d(P, L) = |(P - Q) \cdot \vec{n}|/|\vec{n}|$$

is the distance between P and the plane.

You recognize that this is just the scalar projection of the vector $\vec{QP} = P - Q$ onto the vector $\vec{n}$.

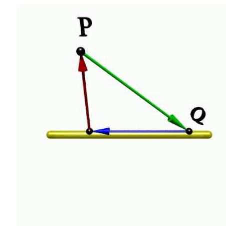


DISTANCE POINT-LINE (3D). If P is a point in space and L is the line $\vec{r}(t) = Q + t\vec{u}$, then

$$d(P, L) = |(P - Q) \times \vec{u}|/|\vec{u}|$$

is the distance between P and the line L.

This formula is verified by writing $(P - Q) \times \vec{u} = |P - Q||\vec{u}| \sin(\theta)$.

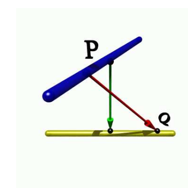


DISTANCE LINE-LINE (3D). L is the line $\vec{r}(t) = Q + t\vec{u}$ and M is the line $\vec{s}(t) = P + t\vec{v}$, then

$$d(L, M) = |(P - Q) \cdot (\vec{u} \times \vec{v})|/|\vec{u} \times \vec{v}|$$

is the distance between the two lines L and M.

This formula is just the scalar projection of $\vec{QP} = P - Q$ onto the vector $\vec{n} = \vec{u} \times \vec{v}$ normal to both $\vec{u}$ and $\vec{v}$.



PLANE THROUGH 3 POINTS P, Q, R: The vector $(a, b, c) = \vec{n} = (Q - P) \times (R - P)$ is normal to the plane. Therefore, the equation is $ax + by + cz = d$. The constant is $d = ax_0 + by_0 + cz_0$ because the point $P = (x_0, y_0, z_0)$ is on the plane.

PLANE THROUGH POINT P AND LINE $\vec{r}(t) = Q + t\vec{u}$. The vector $(a, b, c) = \vec{n} = \vec{u} \times (Q \vec{-} P)$ is normal to the plane. Therefore the plane is given by $ax + by + cz = d$, where $d = ax_0 + by_0 + cz_0$ and $P = (x_0, y_0, z_0)$.

LINE ORTHOGONAL TO PLANE ax+by+cz=d THROUGH POINT P. The vector $\vec{n} = (a, b, c)$ is normal to the plane. The line is $\vec{r}(t) = P + \vec{n}t$.

ANGLE BETWEEN PLANES. The angle between the two planes $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$ is $\arccos(\vec{n}_1 \cdot \vec{n}_2/(|\vec{n}_1||\vec{n}_2|)$, where $\vec{n}_i = (a_i, b_i, c_i)$. Alternatively, it is $\arcsin(|\vec{n}_1 \times \vec{n}_2|/(|\vec{n}_1||\vec{n}_2|))$.

INTERSECTION BETWEEN TWO PLANES. Find the line which is the intersection of two non-parallel planes $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$. Find first a point P which is in the intersection. Then $\vec{r}(t) = P + t(\vec{n}_1 \times \vec{n}_2)$ is the line, we were looking for.

LINES IN THE PLANE.

LINE $P = Q + t\vec{v}$. Eliminating t gives a single equation. For example, $(x, y) = (1, 2) + t(3, 4)$ is equivalent to $x = 1 + 3t, y = 2 + 4t$ and $4x - 3y = -2$. The general equation for a line in the plane is $\boxed{ax + by = d}$.