LINE INTEGRALS.

$$\int_C F \cdot dr = \int_a^b F(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$$

is called the **line integral** of F along the curve C. If F is a force, then the line integral is work.

A RIDDLE ABOUT WORK.

1) If you accelerate a car from $0m/s$ to $10m/s = 2.24$miles/hour the kinetic energy needed is mass times $10^2/2 = 50$.

2) If you accelerate a car from $10m/s$ to $20m/s = 4.48$miles/hour the kinetic energy has increased by $20^2/2 - 10^2/2 = 200 - 50 = 150$.

3) Now watch situation 2) from a moving coordinate system which moves with constant $10m/s$. In that system, the car accerates from 0 to 10 meter per second.

Because all physical laws are the same in different coordinate systems (going from one to the other is a Galilei transformation), the energy should be the same when accelerating from 0 to 10 or from 10 to 20.

This is in contradiction to the fact that accelerating the car from 0 to 10 needs three times less energy than accelerating the car from 10 to 20. Why?

The picture above shows the "Smart car",
an extremly energy friendly car seen often
in Europe, where gas prizes are much higher
than here (even now)

EXAMPLE. Compute the line integral for the vector field $F(x,y) = (x^2, y^2)$ along the boundary of a square in the counter clockwise direction.

We split up the path into four paths:
$\gamma_1 : r(t) = (t, 0)$ for $t \in [0,1]$, $\gamma_2 : r(t) = (1, t)$ for $t \in [0,1]$,
$\gamma_3 : r(t) = (1-t, 1)$ for $t \in [0,1]$, $\gamma_4 : r(t) = (0, 1-t)$ for $t \in [0,1]$,
then $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$.

The integral is then

$$\int_\gamma F \cdot dr = \int_{\gamma_1} F \cdot dr + \int_{\gamma_2} F \cdot dr + \int_{\gamma_3} F \cdot dr + \int_{\gamma_4} F \cdot dr$$
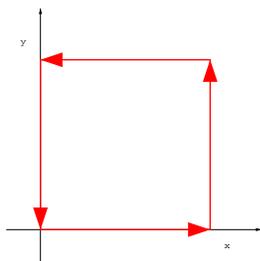
Plugging in $F \cdot dr = F(r(t)) \cdot r'(t)$ gives

$$\int_0^1 (t^2, 0) \cdot (1,0) \, dt + \int_0^1 (1, t^2) \cdot (0,1) \, dt + \int_0^1 ((1-t)^2, 1) \cdot (-1, 0) \, dt + \int_0^1 (0, (1-t)^2) \cdot (0, -1) \, dt$$

and end up with $\int_0^1 2t^2 - 2(1-t)^2 \, dt = 0$.

AMPERES LAW. A wire along the z-axes carries a current I. What is the strength $|B|$ of the magnetic field $\vec{B}(x,y,z) = (-By, Bx, 0)$ in distance r from the z-axes if we know that Amère law

$$\int_C \vec{B} \cdot \vec{dr} = \mu_0 I$$

holds for a closed curve $C : \vec{r}(t) = (r\cos(t), r\sin(t), 0)$ winding in distance r once around the x axes and where μ_0 is a constant. Remark. The Ampere law actually follows from a basic identity which is one of the Maxwell equations and which we will see later in this course.
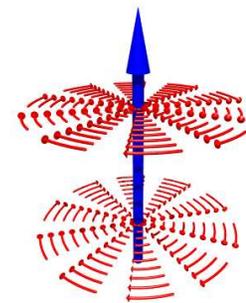
Solution: compute $B(r(t)) = (-B\sin(t), B\cos(t), 0)$ and $r'(t) = (-r\sin(t), r\cos(t))$ The line integral is $2\pi r B$ and because we know this is μI, we have $B = \mu I/2\pi r$.

We see that the magnetic field decays like $1/r$. One can actually derive this result also from special relativity: A striking application of the Lorentz transformation is that if you take two wires and let an electric current flow in the same direction, then the distance between the electrons shrinks: the positively charged ions in the wire see a larger electron density than the ion density. The other wires appears negatively "charged" for the ions and attract each other. If the currents flow in different directions and we go into a coordinate system, where the electrons are at rest in the first wire, then the ion density of the ions in the same wire appears denser and also the electron density is denser in the other wire. The two wires then repell each other. The force is proportional to $1/r$ because two charged wires in distance r attract or repel each other with the force $1/r$.
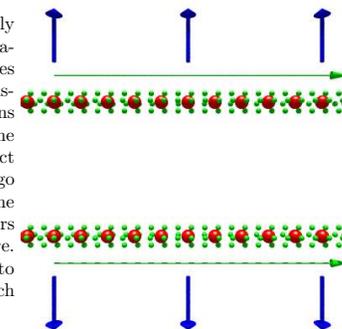
A PERPETUUM MOTION MACHINE. Before entering high school, one of my passions was to construct "perpetuum motion machines". Physics classes in school quickly killed that dream. But still, these machines still fascinate me, especially if they work! Will will see more about these machines next week, but here is a model which allows us to practice line integrals:

Problem: by arranging cleverly charged plates (see lecture), we realize a static electric field $\vec{E} = (0, 0, x)$. We construct a wire along the elliptical path $C : r(t) = (\cos(t), 0, 3\sin(t))$. What is the Voltage

$$V = \int_C \vec{E}(r(t)) \cdot r'(t) \, dt$$

we measure at the wire?

Answer: $\int_0^{2\pi} 3\cos^2(t) \, dt = 3\pi$.

30 second background info in electricity: when an electron is moved around in an electric field along a path C, it gains the potential $\int_C \vec{E} \cdot \vec{dr}$. This potential is also called "voltage" and measured in "Volts". When moving a charge through a voltage difference, it gains some energy. This energy is proportional to the amount of charge going through the wire as well as the voltage V. If the charge is It which is "current" times "time", then the energy is VIt, which can also be seen as the power VI ("volt times ampere = Watts") multiplied with time. For example, through a light bulb of $100W$, a current of about one ampere flows if the voltage is 110 Volts. If we let the lamp burn for 10 hours, we use the energy $10 \cdot 100Wh$ which is one Watt hour. In Massachusetts, a Kilowatt hour costs about $10 - 15$ cents.

Running a 100 Watt light bulb for 24 hours costs a quarter.