

## LINE INTEGRALS

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### LINE INTEGRALS.

2D: If  $F(x, y)$  is a vector field in the plane and  $\gamma : t \mapsto \vec{r}(t)$  is a curve, then  $\int_a^b F(\vec{r}(t)) \cdot \vec{r}'(t) dt$  is called the **line integral** of  $F$  along the curve  $\gamma$ .

3D: If  $F(x, y, z)$  is a vector field in space and  $\gamma : t \mapsto \vec{r}(t)$  is a curve, then  $\int_a^b F(\vec{r}(t)) \cdot \vec{r}'(t) dt$  is called the **line integral** of  $F$  along the curve  $\gamma$ .

NOTATION. The short-hand notation  $\int_\gamma F \cdot ds$  is also used. In the literature, where curves are sometimes written as  $r(t) = (x(t), y(t), z(t))$  or  $r(t)$ , the notation  $\int_\gamma F \cdot dr$  or  $\int_\gamma F \cdot dr$  appears. For simplicity, we leave out below the arrows above the  $r(t)$  and  $F(r(t))$  even so they are vectors.

WRITTEN OUT. If  $F = (M, N)$  and  $\vec{r} = (x(t), y(t))$ , we can write  $\int_a^b M(x(t), y(t))x'(t) + N(x(t), y(t))y'(t) dt$ .

MORE NOTATION. One also can write  $\int_a^b M(x, y) dx + N(x, y) dy$ . Warning: this later notation is only possible for certain type of curves and rather old fashioned. Avoid it.

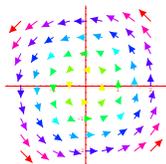
EXAMPLE: **Work**. If  $F(x, y, z)$  is a force field, then the line integral  $\int_a^b F(r(t)) \cdot r'(t) dt$  is called **work**.

EXAMPLE: **Electric potential**. If  $E(x, y, z)$  is an electric field, then the line integral  $\int_a^b E(r(t)) \cdot r'(t) dt$  is called **electric potential**.

EXAMPLE: **Gradient field**. If  $F(x, y, z) = \nabla U(x, y, z)$  is a gradient field, then as we will see next hour  $\int_a^b F(r(t)) \cdot r'(t) dt = U(r(b)) - U(r(a))$ . The gradient field has physical relevance. For example, if  $U(x, y, z)$  is the pressure distribution in the atmosphere, then  $\nabla U(x, y, z)$  is the pressure gradient roughly the wind velocity field.

EXAMPLE 1. Let  $\gamma : t \mapsto r(t) = (\cos(t), \sin(t))$  be a circle parametrized by  $t \in [0, 2\pi]$  and let  $F(x, y) = (-y, x)$ . Calculate the line integral  $I = \int_\gamma F(r) \cdot dr$ .

ANSWER: We have  $I = \int_0^{2\pi} F(r(t)) \cdot r'(t) dt = \int_0^{2\pi} (-\cos(t), \sin(t)) \cdot (-\cos(t), \sin(t)) dt = \int_0^{2\pi} \cos^2(t) + \sin^2(t) dt = 2\pi$



EXAMPLE 2. Let  $r(t)$  be a curve given in polar coordinates as  $r(t) = \cos(t), \phi(t) = t$  defined on  $[0, \pi]$ . Let  $F$  be the vector field  $F(x, y) = (-xy, 0)$ . Calculate the line integral  $\int_\gamma F \cdot dr$ .

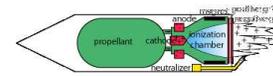
SOLUTION. In Cartesian coordinates, the curve is  $r(t) = (\cos^2(t), \cos(t) \sin(t))$ . The velocity vector is then  $r'(t) = (-2 \sin(t) \cos(t), -\sin^2(t) + \cos^2(t)) = (x(t), y(t))$ . The line integral is

$$\begin{aligned} \int_0^\pi F(r(t)) \cdot r'(t) dt &= \int_0^\pi (\cos^4(t) \sin(t), 0) \cdot (-2 \sin(t) \cos(t), -\sin^2(t) + \cos^2(t)) dt \\ &= -2 \int_0^\pi \sin^2(t) \cos^4(t) dt = -2(t/16 + \sin(2t)/64 - \sin(4t)/64 - \sin(6t)/192)|_0^\pi = -\pi/8 \end{aligned}$$

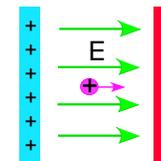
WORK. If  $F$  is a **force field** and  $r(t)$  a path of a body, then  $F(r(t))$  is the force acting on the body. The component of that force in the velocity direction is  $G(t) = F(r(t)) \cdot r'(t)/|r'(t)|$ . For some small time  $dt$ , the body will move a distance  $|r'(t)|dt$ . In physics,  $G(t)ds$  is the amount of work done when traveling this distance. Integrating up gives the total work or energy  $W = \int_a^b G(t)|r'(t)| dt = \int_a^b F(r(t)) \cdot r'(t) dt$ .

$W = \int_\gamma F \cdot ds$  is the **energy** gained by a body traveling along the path  $\gamma$  in a force field  $F$ .

EXAMPLE. In **ion rockets** (used for example in "deep space" space craft), ionized xenon gas is passed by an electrically charged plate and accelerated to high velocities 30km/s. This (kinetic) energy is built up as work in a force field  $F$  which is parallel to an electric field  $E$ . Let  $\gamma : r(t) = (t, 0, 0), 0 \leq t \leq L$  be the path a particle travels between the positively charged plate and the negatively charged plate.



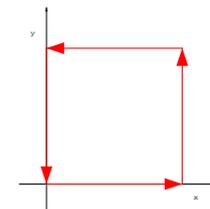
$\int_\gamma E \cdot dr$  is the voltage difference between the two plates and  $\int_\gamma F \cdot dr$  is the energy difference of the particle. Because  $F = cE$ , where  $e$  is the charge of the ion and the velocity  $r'$  of the ions is parallel to the field, we know that  $ELc$  is the voltage difference and  $FL$  is the energy difference which is  $mv^2/2$ , where  $m$  is the mass and  $v$  the velocity of the ion, we could get the electric field strength  $E = mv^2/(2Lc)$ .



### ADDING AND SUBTRACTING CURVES.

If  $\gamma_1, \gamma_2$  are curves, then  $\gamma_1 + \gamma_2$  denotes the curve obtained by traveling first along  $\gamma_1$ , then along  $\gamma_2$ . One writes  $-\gamma$  for the curve  $\gamma$  traveled backwards and  $\gamma_1 - \gamma_2 = \gamma_1 + (-\gamma_2)$ .

EXAMPLES. If  $\gamma_1(t) = (t, 0)$  for  $t \in [0, 1]$ ,  $\gamma_2(t) = (1, (t-1))$  for  $t \in [1, 2]$ ,  $\gamma_3(t) = (1 - (t-2), 1)$  for  $t \in [2, 3]$ ,  $\gamma_4(t) = (0, 1 - (t-3))$  for  $t \in [3, 4]$ , then  $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$  for  $t \in [0, 4]$  is the path which goes around the unit square. The path  $-\gamma$  travels around in the clockwise direction.

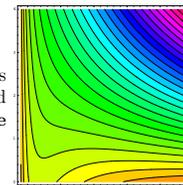


### CALCULATING WITH LINE-INTEGRALS.

- $\int_\gamma F \cdot dr + \int_\gamma G \cdot dr = \int_\gamma (F + G) \cdot dr.$
- $\int_{\gamma_1 + \gamma_2} F \cdot dr = \int_{\gamma_1} F \cdot dr + \int_{\gamma_2} F \cdot dr$
- $\int_\gamma cF \cdot dr = c \int_\gamma F \cdot dr.$
- $\int_{-\gamma} F \cdot dr = - \int_\gamma F \cdot dr.$

### VOLUME-PRESSURE.

Processes involving gases or liquids can be described in a **Volume-pressure diagram**.



Left: V-P diagram  
Right: Sadi Carnot



A periodic processes like a refrigerator defines a closed cycle  $\gamma : t \mapsto r(t) = (V(t), p(t))$  in the  $V - p$  plane. The curve is parameterized by the time  $t$ . At a given time the gas has a specific volume  $V(t)$  and a specific pressure  $p(t)$ . Consider the vector field  $F(V, p) = (p, 0)$  and a closed curve  $\gamma$  and the line integral  $\int_\gamma F \cdot ds$ . Writing it out, we get  $\int_0^{2\pi} (p(t), 0) \cdot (V'(t), p'(t)) dt = \int_0^{2\pi} p(t)V'(t) dt = \int_0^{2\pi} p dV$ .

If the volume of the gas changes under pressure  $p$ , then the work on the system is  $pdV$ . On the other hand, if the volume is kept constant, then for a gas, one does not do work on the system, when changing the pressure. Processes described by this approximation are called **adiabatic**.

For example, if the volume is decreased under high pressure and increased under low pressure then we do the work  $\int_0^{2\pi} p dV$ . Lets compute that if  $r(t) = (2 + \cos(t), 2 + \sin(t))$  for  $t \in [0, 2\pi]$  and  $F(V, p) = (p, 0)$ .  $r'(t) = (-\sin(t), \cos(t))$ ,  $F(r(t)) = (2 + \sin(t), 0)$ . so that  $F(r(t)) \cdot r'(t) = -\sin(t)(2 + \sin(t))$  and  $\int_0^{2\pi} F(r(t)) \cdot r'(t) dt = -\int_0^{2\pi} \sin^2(t) dt = \pi$ .