FUNCTIONS I — Maths21a, O. Knill

FUNCTIONS, DOMAIN AND RANGE. We deal with functions $f(x,y)$ of two variables defined on a **domain** D. The domain is usually the entire plane like for $f(x,y) = x^2 + \sin(xy)$. But there are cases like in $f(x,y) = 1/\sqrt{1-(x^2+y^2)}$, where the domain is a subset of the plane. The **range** is the set of possible values of f.

EXAMPLES.

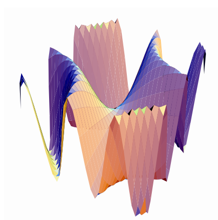
function $f(x,y)$	domain D	range $f(D)$
$f(x,y) = \sin(3x+3y) - \log(1-x^2-y^2)$	open unit disc $x^2+y^2<1$	$[-1,\infty)$
$f(x,y) = f(x,y) = x^2+y^3-xy+\cos(xy)$	entire plane R^2	entire real line
$f(x,y) = \sqrt{4-x^2-2y^2}$	closed elliptic region $x^2+2y^2 \leq 4$	$[0,2]$
$f(x,y) = 1/(x^2+y^2-1)$	everything but the unit circle	entire real line
$f(x,y) = 1/(x^2+y^2)^2$	everything but the origin	positive real axis

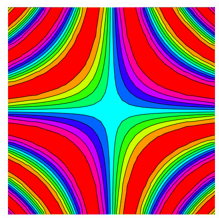
LEVEL CURVES
If $f(x,y)$ is a function of two variables, then $f(x,y) = c = const$ is a **curve** or a collection of curves in the plane. It is called **contour curve** or **level curve**. For example, $f(x,y) = 4x^2 + 3y^2 = 1$ is an ellipse. Level curves allow to visualize functions of two variables $f(x,y)$.

LEVEL SURFACES. We will later see surfaces which are the 3D analog of level curves. if $f(x,y,z)$ is a function of three variables and c is a constant then $f(x,y,z) = c$ is a surface in space. It is called a **contour surface** or a **level surface**. For example if $f(x,y,z) = 4x^2+3y^2+z^2$ then the contour surfaces are ellipsoids. We will come to that tomorrow.

EXAMPLE. Let $f(x,y) = x^2 - y^2$. The set $x^2 - y^2 = 0$ is the union of the sets $x = y$ and $x = -y$. The set $x^2 - y^2 = 1$ consists of two hyperbola with with their tips at $(-1,0)$ and $(1,0)$. The set $x^2 - y^2 = -1$ consists of two hyperbola with their tips at $(0,\pm 1)$.

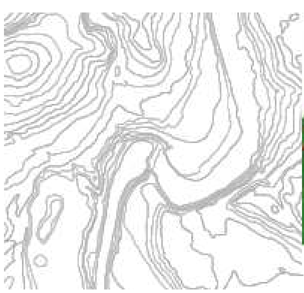
CONTOUR MAP. Drawing several contour curves $\{f(x,y) = c\}$ or several produces what one calls a **contour map**.



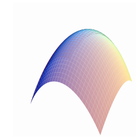


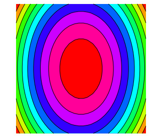
The example shows the graph of the function $f(x,y) = \sin(xy)$. We draw the contour map of f: The curve $\sin(xy) = c$ is $xy = C$, where $C = \arcsin(c)$ is a constant. The curves $y = C/x$ are hyperbolas except for $C = 0$, where $y = 0$ is a line. Also the line $x = 0$ is a contour curve. The contour map is a family of hyperbolas and the coordinate axis.

TOPOGRAPHY. Topographical maps often show the curves of equal height. With the contour curves as information, it is usually already possible to get a good picture of the situation.



EXAMPLE. $f(x,y) = 1 - 2x^2 - y^2$. The contour curves $f(x,y) = 1 - 2x^2 + y^2 = c$ are the ellipses $2x^2 + y^2 = 1 - c$ for $c < 1$.



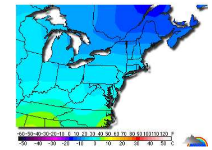


SPECIAL LINES. Level curves are encountered every day:

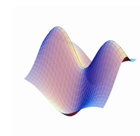
Isobars:	pressure
Isoclines:	direction

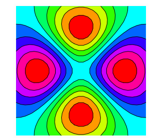
Isothermes:	temperature
Isoheight:	height

For example, the isobars to the right show the lines of constant temperature in the north east of the US.



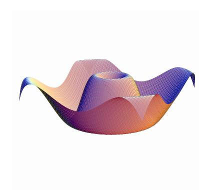
A SADDLE. $f(x,y) = (x^2 - y^2)e^{-x^2-y^2}$. We can here no more find explicit formulas for the contour curves $(x^2-y^2)e^{-x^2-y^2} = c$. Lets try our best:

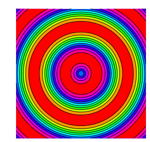




- $f(x,y) = 0$ means $x^2 - y^2 = 0$ so that $x = y, x = -y$ are contour curves.
- On $y = ax$ the function is $g(x) = (1-a^2)x^2 e^{-(1+a^2)x^2}$.
- Because $f(x,y) = f(-x,y) = f(x,-y)$, the function is symmetric with respect to reflections at the x and y axis.

A SOMBRERO. The surface $z = f(x,y) = \sin(\sqrt{x^2+y^2})$ has circles as contour lines.





ABOUT CONTINUITY. In reality, one sometimes has to deal with functions which are not smooth or not continuous: For example, when plotting the temperature of water in relation to pressure and volume, one experiences **phase transitions**, an other example are water waves breaking in the ocean. Mathematicians have also tried to explain "catastrophic" events mathematically with a theory called "catastrophe theory". Discontinuous things are useful (for example in switches), or not so useful (for example, if something breaks).

DEFINITION. A function $f(x,y)$ is **continuous** at (a,b) if $f(a,b)$ is finite and $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. The later means that that along any curve $\vec{r}(t)$ with $r(0) = (a,b)$, we have $\lim_{t\to 0} f(\vec{r}(t)) = f(a,b)$. Continuity for functions of more variables is defined in the same way.

EXAMPLE. $f(x,y) = (xy)/(x^2+y^2)$. Because $\lim_{(x,x)\to(0,0)} f(x,x) = \lim_{x\to 0} x^2/(2x^2) = 1/2$ and $\lim_{(x,0)\to(0,0)} f(0,x) = \lim_{(x,0)\to(0,0)} 0 = 0$. The function is not continuous.

EXAMPLE. $f(x,y) = (x^2 y)/(x^2+y^2)$. In polar coordinates this is $f(r,\theta) = r^3\cos^2(\theta)\sin(\theta)/r^2 = r\cos^2(\theta)\sin(\theta)$. We see that $f(r,\theta) \to 0$ uniformly if $r \to 0$. The function is continuous.