

This is part 2 (of 2) of the weekly homework. It is due August 3 at the beginning of class.

SUMMARY. $dA = dx dy = r dr d\theta$ area element.

- $\int \int_R f(x, y) dx dy = \int_{\alpha}^{\beta} \int_a^b f(r \cos(\theta), r \sin(\theta)) r dr d\theta$
integral in polar coordinates.
- $\int \int_R f(x, y) dx dy / \int \int_R 1 dx dy$ is the **average value** of f on R .
- A curve $\vec{r}(t) = (f(t) \cos(t), f(t) \sin(t))$ can in polar coordinates (r, θ) be given as $r(\theta) = f(\theta)$.
- A vector valued function $\vec{r}(u, v)$ defines a **parametric surface** defined on a region R . It has the **surface area** $\int \int_R |\vec{r}_u(u, v) \times \vec{r}_v(u, v)| du dv$.

Homework Problems

- 1) (4 points) Integrate $f(x, y) = x^2$ over the unit disc $\{x^2 + y^2 \leq 1\}$ in two ways, first using Cartesian coordinates, then using polar coordinates.

Solution:

The integral in Cartesian coordinates goes less smooth because we have to compute a 1D integral with partial integration: $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2 dy dx = \int_{-1}^1 2x^2 \sqrt{1-x^2} dx = \pi/4$.

The integral in polar coordinates is easier to get: using the substitution $x = \cos(u)$ we obtain

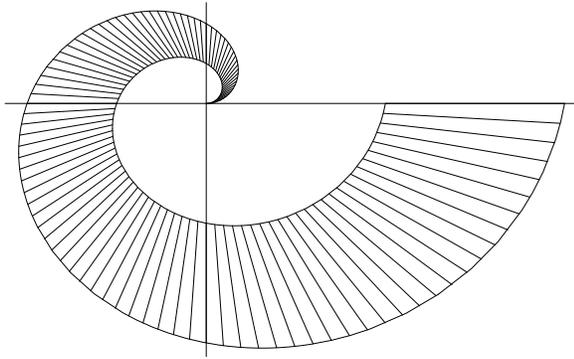
$$\int_0^{2\pi} \int_0^1 r^2 \cos(\theta)^2 r dr d\theta = (1/4)\pi .$$

- 2) (4 points) Find $\int \int_R (x^2 + y^2)^{10} dA$, where R is the part of the unit disc $\{x^2 + y^2 \leq 1\}$ for which $y > x$.

Solution:

Easy in polar coordinates: $\int_0^1 \int_{\pi/4}^{5\pi/4} r^{21} d\theta dr = \pi/22$. This integral would be quite terrible to solve in Cartesian coordinates.

- 3) (4 points) What is the area of the region which is bounded by three curves, first by the polar curve $r(\theta) = \theta$ with $\theta \in [0, 2\pi]$, second by the polar curve $r(\theta) = 2\theta$ with $\theta \in [0, 2\pi]$ and third by the positive x -axes.



Solution:

$$\int_0^{2\pi} \int_{\theta}^{2\theta} r \, dr d\theta = \int_0^{2\pi} (2\theta)^2/2 - (\theta)^2/2 \, d\theta = (3/2) \int_0^{2\pi} \theta^2 \, d\theta = (3/2)(2\pi)^3/3 = 4\pi^3.$$

- 4) (4 points) Find the average value of $f(x, y) = x^2 + y^2$ on the annulus $1 \leq |(x, y)| \leq 2$.

Solution:

The integral of f over the annulus is $2\pi \int_1^2 r^3 \, dr d\theta = 2\pi(16-1)/4$. The area is $2\pi \int_1^2 r \, dr = 2\pi(4-1)/2$. The average is $(15/4)/(3/2) = 5/2$.

- 5) (4 points) Find the surface area of the paraboloid $x = y^2 + z^2$ that lies inside the cylinder $y^2 + z^2 = 9$.

Solution:

We use polar coordinates in the yz -plane. The paraboloid is parametrized by $(u, v) \mapsto (v, v^2 \cos(u), v^2 \sin(u))$ and the surface integral $\int_0^3 \int_0^{2\pi} |\vec{r}_u \times \vec{r}_v| \, dudv$ is equal to $\int_0^3 \int_0^{2\pi} v\sqrt{1+4v^2} \, dudv = 2\pi \int_0^3 v\sqrt{1+4v^2} \, dv = \pi(37^{3/2} - 1)/6$.

Challenge Problems

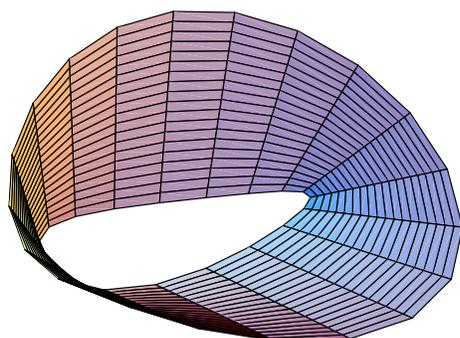
(Solutions to these problems are **not** turned in with the homework.)

- 1) The Moebius strip is a surface which has only one side. It is parametrized as $(1 + (v - 1/2) \cos(u/2)) \cos(u), (1 + (v - 1/2) \sin(u/2)) \sin(u), (v - 1/2) \sin(u/2)$. What surface do you compute with the integral

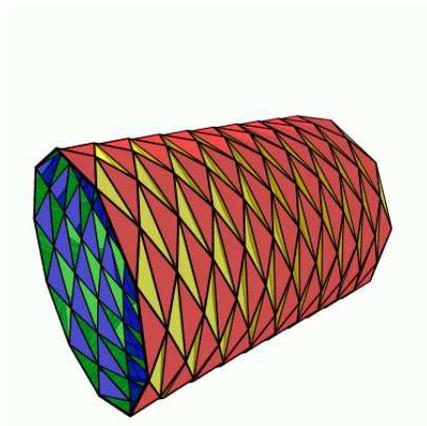
$$\int_0^{2\pi} \int_{-1}^1 |\vec{r}_u(u, v) \times \vec{r}_v(u, v)| \, dudv ?$$

What surface do you compute with the integral

$$\int_0^{4\pi} \int_{-1}^1 |\vec{r}_u(u, v) \times \vec{r}_v(u, v)| \, dudv ?$$



- 2) In class, you have seen a surface which incloses a finite volume and has infinite surface area. Can you construct for any constant M like $M = 10^{100} \text{cm}^2$ a surface inside the unit ball such that the surface area is bigger than M ? The picture below should be a hint.



3) Calculate $\iint_{\mathbf{R}^2} e^{-x^2-y^2} dx dy$ and use this to calculate the integral $\int_{-\infty}^{\infty} e^{-x^2} dx$.

Hint. The function $f(x) = e^{-x^2}$ is known to have no anti-derivative which can be expressed with "known functions" like exp, log, sin etc. You can nevertheless find a closed solution for the definite integral $\int_{-\infty}^{\infty} e^{-x^2} dx$.

4) Find the area of the region shaded in the picture. The region is bounded by the polar curves $r(\theta) = 2\theta$ with $\theta \in [0, 6\pi]$ and $r(\theta) = 3\theta$ with $\theta \in [0, 4\pi]$.

