7/31/2003 SECOND HOURLY PRACTICE Maths 21a, O. Knill, Summer 2003

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.
- The actual exam might have only 9 questions, where question 1 will count 20 points.

1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		100

Problem 1)	(10 points)

T F (answer: F)

The speed of the curve $\vec{r}(t) = (\cos(t), \sin(t), 3t)$ is $(-\sin(t), \cos(t), 3)$.

Solution:
This is the velocity. The speed is the length of this vector and would be $\sqrt{10}$.

T F (answer: T)

Every smooth function of three variables $f(x, y, z)$ satisfies the partial differential equation $f_{xyz} + f_{yzx} = 2f_{zxy}$.

Solution:
By Clairot's theorem.

T F (answer: F)

If $f_x(x, y) = f_y(x, y)$ for all x, y, then $f(x, y)$ is a constant.

Solution:
The transport equation $f_x = f_y$ is a PDE with solutions like for example $f(x, y) = x + y$. Any function which stays invariant by replacing x with y is a solution: like $f(x, y) = \sin(xy) + x^5 y^5$.

T F (answer: F)

$(1, 1)$ is a local maximum of the function $f(x, y) = x^2 y - x + \cos(y)$.

Solution:
$(1, 1)$ is not even a critical point.

T F (answer: F)

If f is a smooth function of two variables, then the number of critical points of f inside the unit disc is finite.

Solution:
Take $f(x, y) = x^2$ for example. Every point on the y axes $\{x = 0\}$ is a critical point.

T F (answer: F)

The value of the function $f(x, y) = \sin(-x + 2y)$ at $(0.001, -0.002)$ can by linear approximation be estimated as -0.003.

Solution:
The correct approximation would be $f(0, 0) + 0.001(-1) - 0.002(2) = -0.005$.

T F

If $(1,1)$ is a critical point for the function $f(x,y)$ then $(1,1)$ is also a critical point for the function $g(x,y) = f(x^2, y^2)$.

Solution:
If $\nabla f(1,1) = (f_x(1,1), f_y(1,1)) = (0,0)$ then also $\nabla g(1,1) = (f_x(1,1)2x, f_y(1,1)2y) = (0,0)$. Note that the statement would not be true, if we would replace $(1,1)$ say with $(2,2)$ (as in the practice exam).

T F

If the velocity vector $\vec{r}\,'(t)$ of the planar curve $\vec{r}(t)$ is orthogonal to the vector $\vec{r}(t)$ for all times t, then the curve is a circle.

Solution:
$d/dt(\vec{r}(t) \cdot \vec{r}(t)) = \vec{r}(t) \cdot \vec{r}'(t)$.

T F

If the double integral $\int\int_R f(x,y)\ dxdy$ is zero for a continuous function $f(x,y)$ and R is the interior of the unit disc, then $f(x,y) = 0$ for at least one point x, y.

Solution:
Assume the conclusion were false, then either $f(x,y) > 0$ everywhere on the disc, or $f(x,y) < 0$ everywhere on the disc. In both cases, the double integral would not vanish.

T F

The surface area is given by the formula $\int\int_R \vec{r}_u \times \vec{r}_v\ dudv$.

Solution:
Take absolute values $|\vec{r}_u \times \vec{r}_v|$.

T F

The gradient of $f(x,y)$ is normal to the level curves of f.

Solution:
This is a basic and important fact.

T F

If (x_0, y_0) is a maximum of $f(x,y)$ under the constraint $g(x,y) = g(x_0, y_0)$, then (x_0, y_0) is a maximum of $g(x,y)$ under the constraint $f(x,y) = f(x_0, y_0)$.

Solution:
Assume you have a situation f, g, where this is true and where the constraint is $g = 0$, produce a new situation $f, h = -g$, where the first statement is still true but where the extrema of h under the constraint of f is a minimum.

[T] [F] If $\vec{u}$ is a unit vector tangent at (x, y, z) to the level surface of $f(x, y, z)$ then the directional derivative satisfies $D_u f(x, y, z) = 0$.

Solution:
The directional derivative measures the rate of change of f in the direction of u. On a level surface, in the direction of the surface, the function does not change (because f is constant by definition on the surface).

[T] **[F]** In polar coordinates (r, ϕ), the integral $\int_0^{\pi} \int)^1 1 \; dr d\theta$ is half the area of the unit disc.

Solution:
The factor r is missing.

[T] [F] The function $u(x, t) = x^2/2 + t$ satisfies the heat equation $u_t = u_{xx}$.

Solution:
Just differentiate.

[T] [F] The vector $\vec{r}_u - \vec{r}_v$ is tangent to the surface parameterized by $\vec{r}(u, v) = (x(u, v), y(u, v), z(u, v))$.

Solution:
Both vectors $\vec{r}_u$ and $\vec{r}_v$ are tangent to the surface. So also their difference.

[T] **[F]** Fubini's theorem assures that $\int_0^1 \int_0^x f(x, y) \; dydx = \int_0^1 \int_0^y f(x, y) \; dxdy$.

Solution:
Fubini is the corresponding statement, when the bounds of integration are constants, that is if you integrate over a rectangle.

T	F

If $(1,1,1)$ is a maximum of f under the constraints $g(x,y,z) = c, h(x,y,z) = d$, and the Lagrange multipliers satisfy $\lambda = 0, \mu = 0$, then $(1,1,1)$ is a critical point of f.

Solution:
Look at the Lagrange equations. If $\lambda = \mu = 0$, then $\nabla f = (0,0,0)$.

T	F

If $(0,0)$ is a critical point of $f(x,y)$ and the discriminant D is zero but $f_{xx}(0,0) > 0$ then $(0,0)$ can not be a local maximum.

Solution:
If $f_{xx}(0,0) > 0$ then on the x-axes the function $g(x) = f(x,0)$ has a local minimum. This means that there are points close to $(0,0)$ where the value of f is larger.

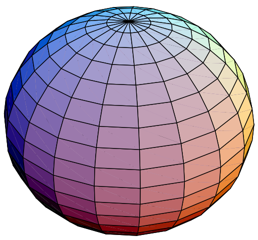
T	F

Let (x_0, y_0) be a saddle point of $f(x,y)$. For any unit vector $\vec{u}$, there are points arbitrarily close to (x_0, y_0) for which ∇f is parallel to $\vec{u}$.

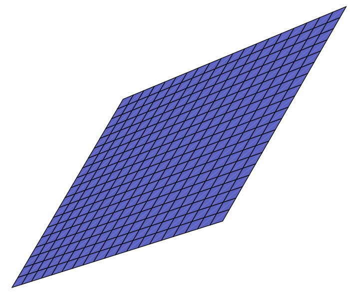
Solution:
Just look at the level curves near a saddle point. The gradient vectors are orthogonal to the level curves which are hyperbola. You see that they point in any direction except 4 directions. To see this better, take a pen and draw a circle around the saddle point between two of your knuckles on your fist. At each point of the circle, now draw the direction of steapest increase (this is the gradient direction).

Problem 2) (10 points)

Match the parametric surfaces $S = \vec{r}(R)$ with the corresponding surface integral $\int\int_S \; dS = \int\int_R |\vec{r}_u \times \vec{r}_v| \; dudv$. No justifications are needed.



I



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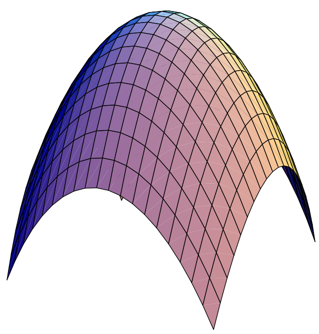
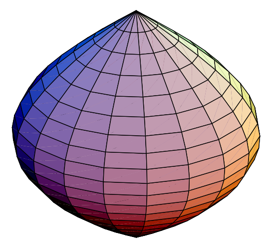


III



IV

Enter I,II,III,IV here	Surface integral
	$\int\int_R |\vec{r}_u \times \vec{r}_v| \; dudv = \int_0^1 \int_0^1 \sqrt{1 + 4u^2 + 4v^2} \; dvdu$
	$\int\int_R |\vec{r}_u \times \vec{r}_v| \; dudv = \int_0^1 \int_0^1 \sqrt{3} \; dvdu$
	$\int\int_R |\vec{r}_u \times \vec{r}_v| \; dudv = \int_0^{2\pi} \int_0^{\pi} \sin(v)\sqrt{1 + \cos(v)^2} \; dvdu$
	$\int\int_R |\vec{r}_u \times \vec{r}_v| \; dudv = \int_0^{2\pi} \int_0^{\pi} \sin(v) \; dvdu$

Solution:

Enter I,II,III,IV here	Surface integral
III	$\int\int_R |\vec{r}_u \times \vec{r}_v| \; dudv = \int_0^1 \int_0^1 \sqrt{1 + 4u^2 + 4v^2} \; dvdu$
II	$\int\int_R |\vec{r}_u \times \vec{r}_v| \; dudv = \int_0^1 \int_0^1 \sqrt{3} \; dvdu$
IV	$\int\int_R |\vec{r}_u \times \vec{r}_v| \; dudv = \int_0^{2\pi} \int_0^{\pi} \sin(v)\sqrt{1 + \cos(v)^2} \; dvdu$
I	$\int\int_R |\vec{r}_u \times \vec{r}_v| \; dudv = \int_0^{2\pi} \int_0^{\pi} \sin(v) \; dvdu$

Problem 3) (10 points)

a) Use the technique of linear approximation to estimate $f(\log(2)+0.001, 0.006)$ for $f(x,y) = e^{2x-y}$. (Here, log means the natural logarithm).
b) Find the equation $ax+by=d$ for the tangent line which goes through the point $(\log(2), 0)$.

Solution:
a) $L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$
$f(x_0,y_0) = e^{2\log 2} = 4$
$f_x(x_0,y_0) = 8$
$f_y(x_0,y_0) = -4$
$L(x,y) = 4 + 0.001 \cdot 8 - 4 \cdot 0.006 = \boxed{3.984}$.
b) We have $a = 8$ and $b = -4$ and get $d = 8\log(2)$ so that the line has the equation $\boxed{8x - 4y = 8\log(2)}$.

Problem 4) (10 points)

a) Show that for any differentiable function $g(x)$, the function $u(x,y) = g(x^2+y^2)$ satisfies the partial differential equation $yu_x = xu_y$.

b) Assuming $g'(5) \neq 0$, let u be the function defined in a). Find the unit vector $\vec{v}$ in the direction of maximal increase at the point $(x,y) = (2,1)$.

Solution:
a) Just differentiate:
$yu_x = yg'(x^2+y^2)2x = 2xyg'(x^2+y^2)$
$xu_y = xg'(x^2+y^2)2y = 2yxg'(x^2+y^2)$
These two expressions are the same.
b) The direction of maximal increase points into the direction of the gradient of u which is $\nabla u(x,y) = (g'(x^2+y^2)2x, g'(x^2+y^2)2y)$.
At the point $(x,y) = (2,1)$ we have $(g'(5)4, g'(5)2)$. If we normalize that, we obtain $\boxed{\vec{v} = (4,2)/\sqrt{20}}$.

Problem 5) (10 points)

Find the point on the surface $g(x,y,z) = \frac{1}{x} + \frac{1}{y} + \frac{8}{z} = 1$ for which the distance to the origin is a local minimum.

Solution:
This is a Lagrange problem. One wants to minimize $f(x,y,z) = x^2 + y^2 + z^2$ under the constraint $g(x,y,z) = 1$. The Lagrange equations are

$$\begin{aligned} \frac{-1}{x^2} &= 2\lambda x \\ \frac{-1}{y^2} &= 2\lambda y \\ \frac{-8}{z^2} &= 2\lambda z \\ \frac{1}{x} + \frac{1}{y} + \frac{8}{z} &= 1 \end{aligned}$$

The first two equations show $x = y$, the first and third equations show $8/z^3 = 1/x^3$ or $z = 2x$. Plugging this into the last equation gives $2/x + 8/(2x) = 1$ or $x = 6, y = 6, z = 12$. $\boxed{(x,y,z) = (6,6,12)}$.
The global picture is interesting: consider the points $(x,y,z) = (1, -1/n, 8/n)$, where n is a large integer, One can check that these points ly on the surface $g(x,y,z) = 1$. Their distance to the origin decreases to 1 if n goes to infinity. So the point $(6,6,12)$, while a local minimum is not a global minimum.

Problem 6) (10 points)

Find all extrema of the function $f(x,y) = x^3 + y^3 - 3x - 12y + 20$ on the plane and characterize them. Do you find a absolute maximum or absolute minimum amoung them?

Solution:
The critical points satisfy $\nabla f(x,y) = (0,0)$ or $(3x^2 - 3, 3y^2 - 12) = (0,0)$. There are 4 critical points $(x,y) = (\pm 1, \pm 2)$. The discriminant is $D = f_{xx} f_{yy} - f_{xy}^2 = 36xy$ and $f_{xx} = 6x$.

point	D	f_{xx}	classification	value
(-1,-2)	72	-6	maximum	38
(-1, 2)	-72	-6	saddle	6
(1, -2)	-72	6	saddle	34
(1, 2)	72	6	minimum	2

Note that there are no global (= absolute) maxima nor global minima because the function takes arbitrarily large and small values. For $y = 0$ the function is $g(x) = f(x,0) = x^3 - 3x + 20$ which satisfies $\lim_{x \to \pm\infty} g(x) = \pm\infty$.

Problem 7) (10 points)

Find

$$\int_0^1 \int_y^1 x^2 e^{xy} \; dxdy \; .$$

Hint. Sketch the region and check the order of integration.

Solution:
Integrate $f(x,y) = x^2 e^{xy}$ over the region, where $0 \leq y \leq 1$ and $y \leq x \leq 1$.
Switch the order of integration: $\int_0^1 \int_0^x x^2 e^{xy} \; dydx = \boxed{(e-1)/2}$.

Problem 8) (10 points)

Integrate the function $f(x,y,z) = z^2$ over the region $\{x^2 + y^2 + z^2 \leq 1\}$.

Solution:
$\int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^4 \cos^2(\phi) \sin(\phi) \; d\rho d\phi d\theta = 2\pi(1/5)(2/3) = \boxed{4\pi/15}$.

Problem 9) (10 points)

Find the surface area of the surface $z = \sqrt{x^2 + y^2} + 1$ that lies above the unit disk $\{x^2 + y^2 \leq 1\}$ in the x-y plane.

Solution:
Write the surface in cylindrical coordinates $r(u,v) = (u\cos[v], u\sin[v], u+1)$. Then $|r_u \times r_v| = \sqrt{2}u$, so that we have

$$\int_0^1 \int_0^{2\pi} \sqrt{2}u \; dvdu = 2\pi\sqrt{2}/2 \; .$$

Problem 10) (10 points)

Let $\vec{r}(t)$ be the space curve $\vec{r}(t) = (t^2, \sin(3\pi t), \cos(5\pi t))$.

a) Calculate the velocity, the acceleration and the speed of $\vec{r}(t)$ at time $t = 1$.
b) Write down the length of the curve from $t = 1$ to $t = 10$ as an integral. You don't have to evaluate the integral.
c) The curve $t \mapsto \vec{r}(t) = (t^3, 1-t, 1-t^3)$ lies in a plane. What is the equation of this plane?

Solution:
a) $\vec{v}(t) = \vec{r}'(t) = (2t, 3\pi\cos(3\pi t), -5\pi\sin(5\pi t))$.
$\vec{v}(1) = (2, -3\pi, 0)$
$\vec{a}(t) = \vec{r}''(t) = (2, -9\pi^2\sin(3\pi t), -25\pi^2\cos(5\pi t))$.
$\vec{a}(1) = (2, 0, 25\pi^2)$. $|v(1)| = \sqrt{4 + 9\pi^2}$.
b) $\int_1^{10} \sqrt{4t^2 + 9\pi^2\cos^2(3\pi t) + 25\pi^2\sin^2(5\pi t)}$.
c) $t = 0 : P = (0, 1, 1), t = 1 : Q = (1, 0, 0), t = 2 : R = (8, -1, -7)$ are points on the Plane. $\vec{PQ} = (1, -1, -1)$, $\vec{PR} = (8, -2, -8)$. Their cross product is $(6, 0, 6)$. The plane is $x + z = 1$.