**7/31/2003 SECOND HOURLY SOLUTIONS Maths 21a, O. Knill, Summer 2003**

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

| | | |
|---|---|---|
| 1 | | 20 |
| max 2a,2b | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 10 |
| 8 | | 10 |
| 9 | | 10 |
| Total: | | 100 |

Problem 1) (20 points)

[✓ T] F $\int_0^1 \int_0^1 x^2 y^5 \, dxdy = 1/18$.

**Solution:**
One can "see" that ...

T [✓ F] If $f_x(x,y) = f_y(x,y) = 0$ for all $(x,y)$ then $f(x,y) = 0$ for all $(x,y)$ .

**Solution:**
False, $f$ could be constant.

T [✓ F] $(0,0)$ is a local maximum of the function $f(x,y) = x^2 - y^2 + x^4 + y^4$.

**Solution:**
$(0,0)$ is a saddle point.

[✓ T] F If the second derivative test shows that $f(x,y)$ has a local maximum at $(0,0)$ then also $g(x,y) = f(x,y) - x^4 + y^3$ has a local maximum at $(0,0)$.

**Solution:**
Adding $x^4 + y^3$ does not change the first and second derivatives.

[✓ T] F The value of the function $f(x,y) = \sqrt{1 + 3x + 5y}$ at $(-0.002, 0.01)$ can by linear approximation be estimated as $1 - (3/2) \cdot 0.002 + (5/2) \cdot 0.01$.

**Solution:**
Use formula for $L(x,y)$.

T [✓ F] Every type I region is also a type II region.

**Solution:**
No way. There are counter examples in the problem after the next problem.

T [✓ F] The directional derivative $D_v f$ is a vector normal to $v$.

**Solution:**
The directional derivative is a number, not a vector.

[✓ T] F The function $y\sin(x-ct)$ satisfies the partial differential equation $f_{tt} = c^2(f_{xx}+f_{yy})$.

**Solution:**
This is a solution of the two dimensional wave equation.

T [✓ F] If a surface bounds a region of finite volume, then its surface area is finite too.

**Solution:**
We have seen in class Gabriel's trumpet.

T [✓ F] The gradient of $f$ at a point $(x_0, y_0, z_0)$ is tangent to the level surface of $f$ which contains $(x_0, y_0, z_0)$.

**Solution:**
It is a basic and important fact that $\nabla f$ is **perpendicular** to the level surface.

T [✓ F] The sign of the Lagrange multiplier $\lambda$ indicates whether the constrained extremum is a local maximum or local minimum.

**Solution:**
No, while changing the sign of $f$ indeed changes the sign of the Lagrange multipliers and also interchanges local maxima and local minima, the change of the sign of $g$ does not change the nature of the critical point but changes the sign of $\lambda$.

[✓ T] F If $D_v f(1,1,1) = 0$ for all vectors $v$, then $(1,1,1)$ is a critical point.

**Solution:**
Especially, $D_{\nabla f}(f) = |\nabla f|^2 = 0$ so that $\nabla f = (0,0,0)$.

T [✓ F] $\int_2^3 \int_{\sin(y)}^{5+e^y} x^2 - y^2 \ dxdy$ is a type I integral.

**Solution:**
It is a type II integral.

T [✓ F] The function $u(x,t) = x^3 + t^3$ satisfies the wave equation $u_{tt} = u_{xx}$.

**Solution:**
Just differentiate.

T [✓ F] The tangent vectors $\vec{r}_u$ and $\vec{r}_v$ to a surface parameterized by $\vec{r}(u,v)$ are always perpendicular to each other.

**Solution:**
This is most of the time wrong, even for parameterizations of planes.

T [✓ F] Every critical point $(x,y)$ of $f(x,y)$ for which the discriminant $D$ is not zero, is either a local maximum, a local minimum or a saddle point.

**Solution:**
This is the second derivative test.

[✓ T] F The function $f(x,y) = e^y x^2 \sin(y^2)$ satisfies the partial differential equation $f_{xxyyyxyy} = 0$.

**Solution:**
By Fubini, we can have all three $x$ derivatives at the beginning.

[✓ T] F If $(0,0)$ is a critical point of $f(x,y)$ and the discriminant $D$ is zero but $f_{xx}(0,0) < 0$ then $(0,0)$ can not be a local minimum.

**Solution:**
If $f_{xx}(0,0) < 0$ then on the x-axes the function $g(x) = f(x,0)$ has a local maximum. This means that there are points close to $(0,0)$ where the value of $f$ is larger.

[✓ T] F In the second derivative test, one can replace in the case $D > 0$ the condition $f_{xx} > 0$ with $f_{yy} > 0$ to check whether a point is a local minimum.

**Solution:**
True. If $f_{xx}f_{yy} - f_{xy}^2 > 0$, then $f_{xx}$ and $f_{yy}$ must have the same signs.

T [✓ F] The integral $\int_0^{2\pi} \int_0^1 1 \, dr d\theta$ is the area of the unit circle.

**Solution:**
The factor $r$ is missing.

| Problem 2a) | (10 points) |
|---|---|

Match the integrals with those obtained by changing the order of integration. No justifications are needed. Note that one of the Roman letters I)-V) will not be used, you have to chose four out of five.

| Enter I,II,III,IV or V here. | Integral |
|---|---|
|  | $\int_0^1 \int_{1-y}^1 f(x,y)\ dxdy$ |
|  | $\int_0^1 \int_y^1 f(x,y)\ dxdy$ |
|  | $\int_0^1 \int_0^{1-y} f(x,y)\ dxdy$ |
|  | $\int_0^1 \int_0^y f(x,y)\ dxdy$ |

I) $\int_0^1 \int_0^x f(x,y)\ dydx$

II) $\int_0^1 \int_0^{1-x} f(x,y)\ dydx$

III) $\int_0^1 \int_x^1 f(x,y)\ dydx$

IV) $\int_0^1 \int_0^{x-1} f(x,y)\ dydx$

V) $\int_0^1 \int_{1-x}^1 f(x,y)\ dydx$

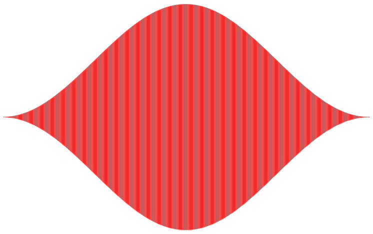
**Solution:**

| Enter I,II,III,IV or V here. | Integral |
|---|---|
| V | $\int_0^1 \int_{1-y}^1 f(x,y)\ dxdy$ |
| I | $\int_0^1 \int_y^1 f(x,y)\ dxdy$ |
| II | $\int_0^1 \int_0^{1-y} f(x,y)\ dxdy$ |
| III | $\int_0^1 \int_0^y f(x,y)\ dxdy$ |

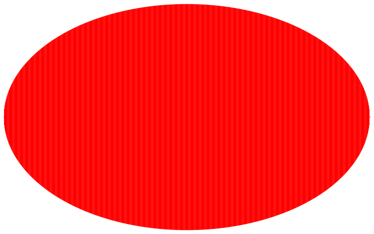
Problem 2b) (10 points)

Check the boxes indicating which of the region is a type I or type II region. (You check both boxes in one row if the region is both type I and type II). In the same table, please match also each region with one of the integrals a)-e). No justifications are needed.

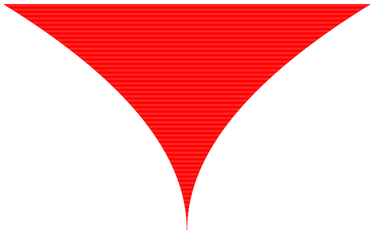
A



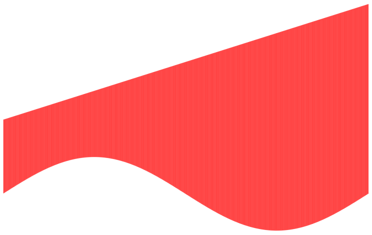
B



C



D



E

| | |
|---|---|
| a) | $\int_0^{2\pi} \int_{\sin(y)}^{2+\cos(y)} f(x,y)\ dxdy$ |
| b) | $\int_0^{2\pi} \int_{\cos(x)}^{2-\cos(x)} f(x,y)\ dydx$ |
| c) | $\int_0^{2\pi} \int_{-y^2}^{y^2} f(x,y)\ dxdy$ |
| d) | $\int_{-2\pi}^{2\pi} \int_{2\sin(x)}^{4+x} f(x,y)\ dydx$ |
| e) | $\int_{-2\pi}^{2\pi} \int_{-\sqrt{4\pi^2-x^2}/2}^{\sqrt{4\pi^2-x^2}/2} f(x,y)\ dydx$ |

| region | type I? check if yes | type II? check if yes | integral enter a)-e) |
|---|---|---|---|
| A | | | |
| B | | | |
| C | | | |
| D | | | |
| E | | | |

**Solution:**

| region | type I? | type II? | integral a)-e) |
|---|---|---|---|
| A | | X | a |
| B | X | X | b |
| C | X | X | e |
| D | X | X | c |
| E | X | | d |

Problem 3) (10 points)

a) Use the technique of linear approximation to estimate $f(0.003, -0.0001, \frac{\pi}{2} + 0.01)$ for

$$f(x,y,z) = \cos(xy + z) + x + 2z\ .$$

b) Find the equation $ax + by + cz = d$ for the tangent plane to the level surface of $f$ at $(0, 0, \frac{\pi}{2})$.

**Solution:**
a) $L(x,y) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x - x_0) + f_y(x_0,y_0,z_0)(y - y_0) + f_z(x_0,y_0,z_0)(z - z_0)$
$f(x_0,y_0,z_0) = \cos(\pi/2) + \pi = \pi$
$a = f_x(x_0,y_0,z_0) = -0\sin(\pi/2) + 1 = 1$
$b = f_y(x_0,y_0,z_0) = -0\sin(\pi/2) = 0$
$c = f_z(x_0,y_0,z_0) = -\sin(\pi/2) + 2 = 1$
$L(x,y,z) = \pi + x + (z - \pi/2)$
$L(0.002, -0.0001, \pi/2 + 0.01) = \pi + 0.003 + 0.01 = \boxed{\pi + 0.013}$.
b) Because $\nabla f(0, 0, \pi/2) = (1, 0, 1)$ the plane is $x + z = \pi/2$, where the right hand side was obtained by plugging in the point $(0, 0, \pi/2)$.

Problem 4) (10 points)

a) Verify that the function $f(x, y) = x^2 - y^2$ satisfies the partial differential equation

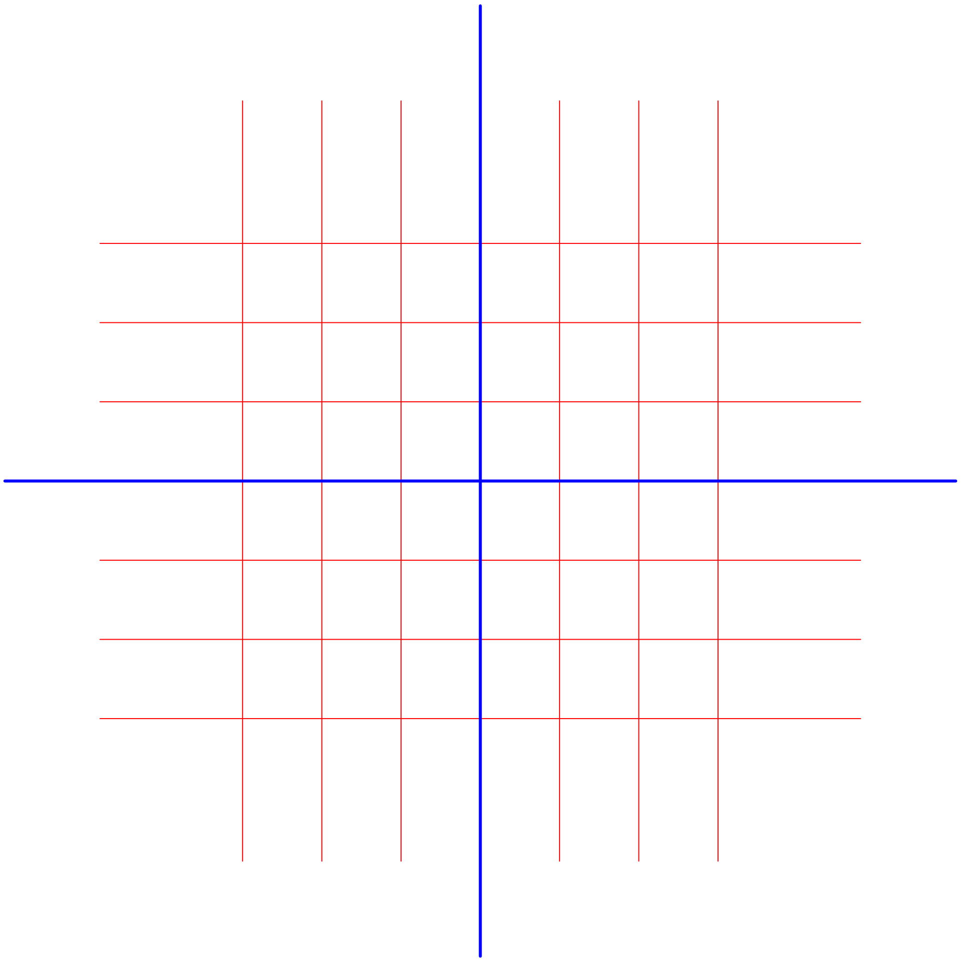
$$x f_x(x, y) + y f_y(x, y) = 2f \; .$$

b) Sketch in the diagram below the level curves $f(x, y) = c$ for values $c = -3, -1, 0, 1, 3$. The distance between two grid lines is 1.

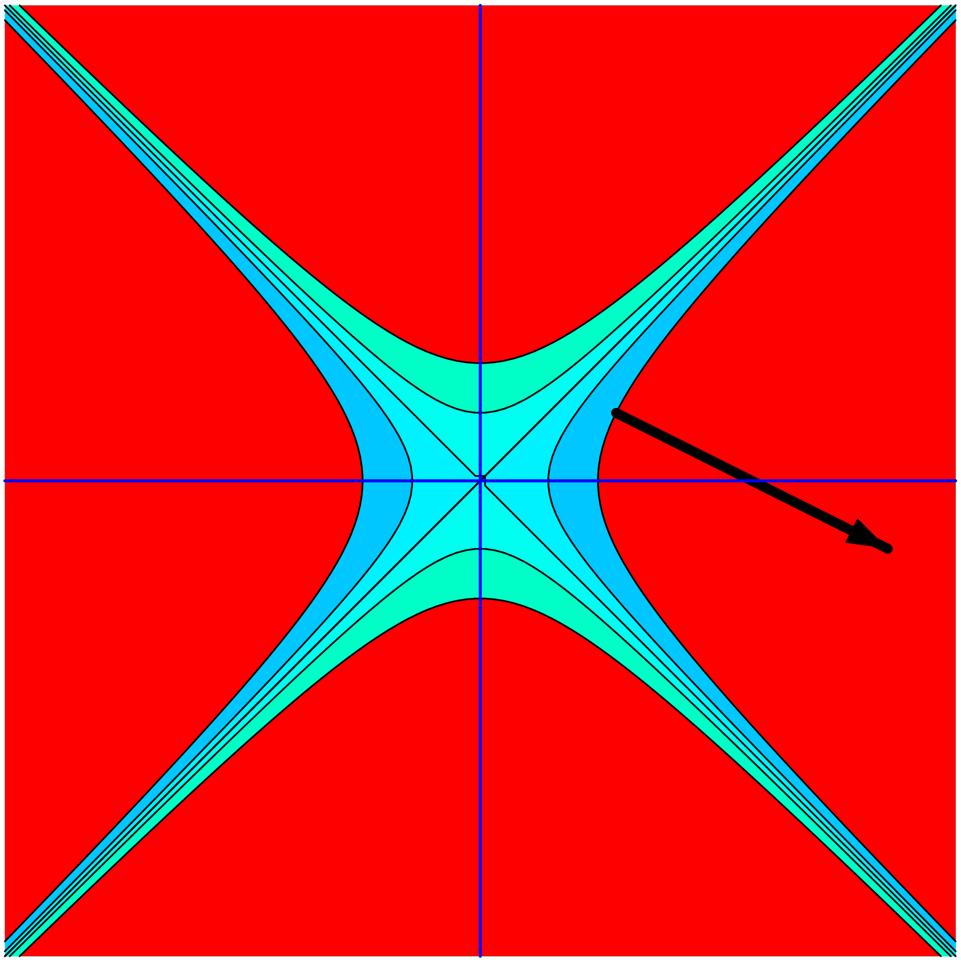
c) Find and draw the gradient vector $\nabla f(x, y)$ at the point $(2, 1)$.

d) Find the equation for the tangent line at the point $(2, 1)$ and draw that line in the picture.

e) Find the directional derivative of $f(x, y)$ at the point $(2, 1)$ into the direction $(1, 0)$.



**Solution:**
a) Direct computation $xf_x + yf_y = 2x^2 - 2y^2 = 2f(x, y)$.
b) The level curves are hyperbola for $c \neq 0$ and two crossing lines for $c = 0$.



c) $\nabla f(x, y) = (2x, -2y)$ so that $\nabla f(2, 1) = \boxed{(4, -2)}$.
d) We know the gradient so that the line has the form $4x - 2y = d$. By plugging in the point $(2, 1)$, we get $d = 6$. The equation is $\boxed{4x - 2y = 6}$.
e) $\nabla f \cdot (1, 0) = \boxed{4}$.

| Problem 5) (10 points) |
|---|

A drawer of length $x$, width $y$ and height $z$ is open on the top and has the volume 32. For which dimensions are the material costs minimal?

Hint. The problem is to minimize $f(x, y, z) = xy + 2yz + 2xz$ under the constraint $g(x, y, z) = xyz = 32$.

**Solution:**
To minimize $f(x, y, z) = xy + 2yz + 2xz$ under the constraint $g(x, y, z) = xyz = 32$ we solve Lagrange equations

$$\begin{aligned} y + 2z &= \lambda yz \\ x + 2z &= \lambda xz \\ 2y + 2x &= \lambda xy \\ xyz &= 32 \end{aligned}$$

Subtracting 2) from the 1) gives $(y - x) = \lambda(y - x)z$. If $y - x$ is not zero, then $\lambda z = 1$ which in 1) would give $y + 2z = y$ or $z = 0$ contradicting 4). Therefore, $y - x = 0$ or $y = x$. Equation 3) gives $4x = \lambda x^2$. Again $x = 0$ would contradict the 4) so that we can divide by $x$ and get $4 = \lambda x$. Equation 2) gives $x + 2z = 4z$ or $x = 2z$. We have now $x = y = 2z$. Using 4) gives $xyz = 4z^3 = 32$ so that $\boxed{z = 2 \text{ and } x = 4,\ y = 4}$.

Problem 6) (10 points)

Find all the critical points of $f(x, y) = \frac{x^5}{5} - \frac{x^2}{2} + \frac{y^3}{3} - y$ and indicate whether they are local maxima, local minima or saddle points.

**Solution:**
$\nabla f(x, y) = (x^4 - x, (y^2 - 1)) = (0, 0)$ so that the critical points are $(0, 1), (0, -1), (1, 1), (1, -1)$. We have $D = (4x^3 - 1)2y$ and $f_{xx} = (4x^3 - 1)$.

| Point | D | $f_{xx}$ | type |
|---|---|---|---|
| $(0, 1)$ | $D = -2$ | $-1$ | saddle point |
| $(0, -1)$ | $D = 2$ | $-1$ | local maximum |
| $(1, 1)$ | $D = 6$ | $3$ | local minimum |
| $(1, -1)$ | $D = -6$ | $3$ | saddle point |

Problem 7) (10 points)

Evaluate the integral $\int_0^\pi \int_x^\pi (2y + \frac{\sin(y)}{y})\ dydx$.

**Solution:**
Change the order of integration: $\int_0^\pi \int_0^y (2y + \frac{\sin(y)}{y})\ dxdy = \int_0^\pi (2y^2 + \sin(y))\ dy = \boxed{2\pi^3/3 + 2}$.

Problem 8) (10 points)

Integrate the function $f(x, y) = y^2$ over the region $\{x^2 + y^2 \leq 1\}$.

**Solution:**
$\int_0^{2\pi} \int_0^1 r^3 \sin^2(\theta) \; d\theta dr = \boxed{\pi/4}$.

| Problem 9) (10 points) |
|---|

Find the surface area of the surface parameterized by $\vec{r}(u,v) = (u+v, u-v, 2u+v)$, where the parameters $(u,v)$ are in the unit disc $u^2+v^2 \leq 1$.

**Solution:**
$\vec{r}_u = (1,1,2)$.
$\vec{r}_v = (1,-1,1)$.
$\vec{r}_u \times \vec{r}_v = (3,1,-2)$
$|\vec{r}_u \times \vec{r}_v| = \sqrt{14}$
$\int\int_R \sqrt{14} \; dA = \boxed{\pi\sqrt{14}}$. This is just $\sqrt{14}$ times the area of the unit disc.