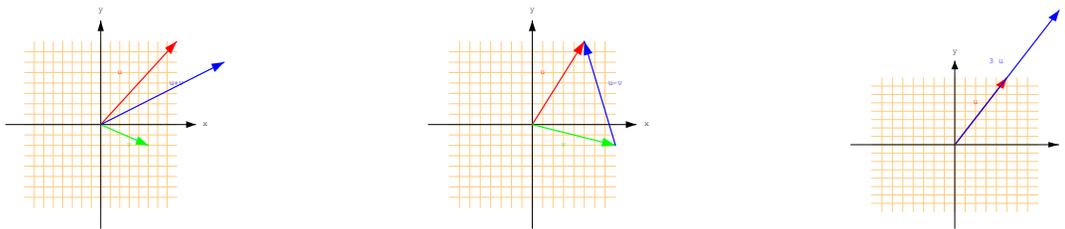


VECTORS. Two points $P_1 = (x_1, y_1, z_1)$, $Q = P_2 = (x_2, y_2, z_2)$ determine a **vector** $\vec{v} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$. It points from P_1 to P_2 and we can write $P_1 + \vec{v} = P_2$.

COORDINATES. Points P in space are in one to one correspondence to vectors pointing from 0 to P . The numbers \vec{v}_i in a vector $\vec{v} = (v_1, v_2, v_3)$ are also called **components** or of the vector.

REMARKS: vectors can be drawn **everywhere** in space. If a vector starts at 0, then the vector $\vec{v} = (v_1, v_2, v_3)$ points to the point (v_1, v_2, v_3) . That's is why one can identify points $P = (a, b, c)$ in space with a vector $\vec{v} = (a, b, c)$. Two vectors which are translates of each other are considered **equal**.

ADDITION SUBTRACTION, SCALAR MULTIPLICATION.



$$\begin{aligned} \vec{u} + \vec{v} &= (u_1, u_2, u_3) + (v_1, v_2, v_3) & \vec{u} - \vec{v} &= (u_1, u_2, u_3) - (v_1, v_2, v_3) & \lambda \vec{u} &= \lambda(u_1, u_2, u_3) \\ &= (u_1 + v_1, u_2 + v_2, u_3 + v_3) & &= (u_1 - v_1, u_2 - v_2, u_3 - v_3) & &= (\lambda u_1, \lambda u_2, \lambda u_3) \end{aligned}$$

BASIS VECTORS. The vectors $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$ and $\vec{k} = (0, 0, 1)$ are called **standard basis vectors**.

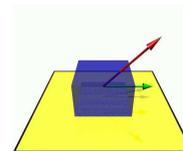
Every vector $\vec{v} = (v_1, v_2, v_3)$ can be written as a sum of standard basis vectors: $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$.

WHERE DO VECTORS OCCUR? Here are some examples:

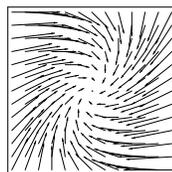
Velocity (see later): if $(f(t), g(t))$ is a point in the plane which depends on time t , then $\vec{v} = (f'(t), g'(t))$ is the **velocity vector** at the point $(f(t), g(t))$.



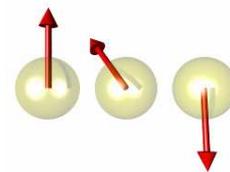
Forces: Some problems in statics involve the determination of a forces acting on objects. Forces are represented as vectors



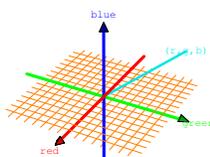
Fields: fields like electromagnetic or gravitational fields or velocity fields in fluids are described with vectors.



Qbits: in quantum computation, one does not work with bits, but with **qbits**, which are vectors.



Color can be written as a vector $\vec{v} = (r, g, b)$, where r is red, g is green and b is blue. An other coordinate system is $\vec{v} = (c, m, y) = (1 - r, 1 - g, 1 - b)$, where c is cyan, m is magenta and y is yellow.



SVG. Scalable Vector Graphics is an emerging standard for the web for describing two-dimensional graphics in XML.



VECTOR OPERATIONS: The addition and scalar multiplication of vectors satisfy "obvious" properties. There is no need to memorize them. We write $*$ here for multiplication with a scalar but usually, the multiplication sign is left out.

$$\begin{aligned} \vec{u} + \vec{v} &= \vec{v} + \vec{u} && \text{commutativity} \\ \vec{u} + (\vec{v} + \vec{w}) &= (\vec{u} + \vec{v}) + \vec{w} && \text{additive associativity} \\ \vec{u} + \vec{0} &= \vec{0} + \vec{u} = \vec{u} && \text{null vector} \\ r * (s * \vec{v}) &= (r * s) * \vec{v} && \text{scalar associativity} \\ (r + s)\vec{v} &= \vec{v}(r + s) && \text{distributivity in scalar} \\ r(\vec{v} + \vec{w}) &= r\vec{v} + r\vec{w} && \text{distributivity in vector} \\ 1 * \vec{v} &= \vec{v} && \text{the one element} \end{aligned}$$

LENGTH. The length $|\vec{v}|$ of \vec{v} is the distance from the beginning to the end of the vector.

EXAMPLES. 1) If $\vec{v} = (3, 4, 5)$, then $|\vec{v}| = \sqrt{50} = 5\sqrt{2}$. 2) $|\vec{i}| = |\vec{j}| = |\vec{k}| = 1$, $|\vec{0}| = 0$.

UNIT VECTOR. A vector of length 1 is called a **unit vector**. If $\vec{v} \neq \vec{0}$, then $\vec{v}/|\vec{v}|$ is a unit vector.

EXAMPLE: If $\vec{v} = (3, 4)$, then $\vec{v} = (2/5, 3/5)$ is a unit vector, $\vec{i}, \vec{j}, \vec{k}$ are unit vectors.

PARALLEL VECTORS. Two vectors \vec{v} and \vec{w} are called **parallel**, if $\vec{v} = r\vec{w}$ with some constant r .

DOT PRODUCT. The **dot product** of two vectors $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$ is defined as

$$\vec{v} \cdot \vec{w} = v_1w_1 + v_2w_2 + v_3w_3$$

Remark: in science, other notations are used: $\vec{v} \cdot \vec{w} = (\vec{v}, \vec{w})$ (mathematics) $\langle \vec{v} | \vec{w} \rangle$ (quantum mechanics) $v_i w^i$ (Einstein notation) $g_{ij} v^i w^j$ (general relativity). The dot product is also called **scalar product**, or **inner product**.

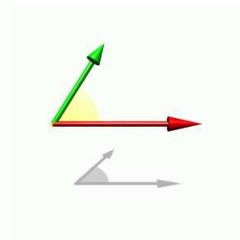
LENGTH. Using the dot product one can express the length of \vec{v} as $|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$.

CHALLENGE. Express the dot product in terms of the length alone.

SOLUTION: $(\vec{v} + \vec{w}, \vec{v} + \vec{w}) = (\vec{v}, \vec{v}) + (\vec{w}, \vec{w}) + 2(\vec{v}, \vec{w})$ can be solved for (\vec{v}, \vec{w}) .

ANGLE. Because $|\vec{v} - \vec{w}|^2 = (\vec{v} - \vec{w}, \vec{v} - \vec{w}) = |\vec{v}|^2 + |\vec{w}|^2 - 2(\vec{v}, \vec{w})$ is by the **cos-theorem** equal to $|\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}| \cdot |\vec{w}| \cos(\alpha)$, where α is the angle between the vectors \vec{v} and \vec{w} , we get the important formula

$$\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha)$$



CAUCHY-SCHWARZ INEQUALITY: $|\vec{v} \cdot \vec{w}| \leq |\vec{v}| |\vec{w}|$ follows from that formula because $|\cos(\alpha)| \leq 1$.

TRIANGLE INEQUALITY: $|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$ follows from $|\vec{u} + \vec{v}|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = \vec{u}^2 + \vec{v}^2 + 2\vec{u} \cdot \vec{v} \leq \vec{u}^2 + \vec{v}^2 + 2|\vec{u}| \cdot |\vec{v}| \leq \vec{u}^2 + \vec{v}^2 + 2|\vec{u}| \cdot |\vec{v}| = (|\vec{u}| + |\vec{v}|)^2$.

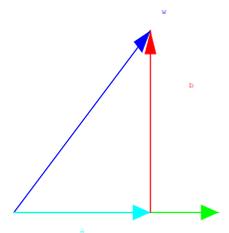
FINDING ANGLES BETWEEN VECTORS. Find the angle between the vectors $(1, 4, 3)$ and $(-1, 2, 3)$.

ANSWER: $\cos(\alpha) = 16/(\sqrt{26}\sqrt{14}) \sim 0.839$. So that $\alpha = \arccos(0.839..) \sim 33^\circ$.

ORTHOGONAL VECTORS. Two vectors are called **orthogonal** if $\vec{v} \cdot \vec{w} = 0$. The zero vector $\vec{0}$ is orthogonal to any vector. EXAMPLE: $\vec{v} = (2, 3)$ is orthogonal to $\vec{w} = (-3, 2)$.

PROJECTION. The vector $\vec{a} = \text{proj}_{\vec{w}}(\vec{v}) = \vec{w}(\vec{v} \cdot \vec{w}/|\vec{w}|^2)$ is called the **projection** of \vec{v} onto \vec{w} .

The **scalar projection** is defined as $\text{comp}_{\vec{w}}(\vec{v}) = (\vec{v} \cdot \vec{w})/|\vec{w}|$. (Its absolute value is the length of the projection of \vec{v} onto \vec{w} .) The vector $\vec{b} = \vec{v} - \vec{a}$ is called the **component** of \vec{v} orthogonal to the \vec{w} -direction.



EXAMPLE. $\vec{v} = (0, -1, 1)$, $\vec{w} = (1, -1, 0)$, $\text{proj}_{\vec{w}}(\vec{v}) = (1/2, -1/2, 0)$, $\text{comp}_{\vec{w}}(\vec{v}) = 1/\sqrt{2}$.