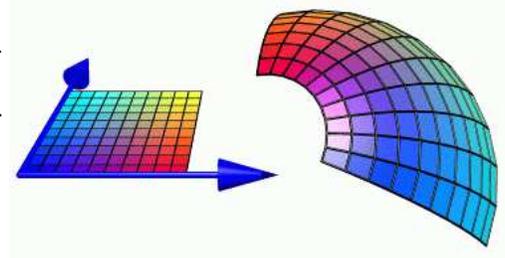


SURFACE AREA

$$\int \int_R |\vec{r}_u(u, v) \times \vec{r}_v(u, v)| \, dudv$$

is the area of the surface.

INTEGRAL OF A SCALAR FUNCTION ON A SURFACE. If S is a surface, then $\int \int_S f(x, y) \, dS$ should be an average of f on the surface. If $f(x, y) = 1$, then $\int \int_S \, dS$ should be the area of the surface. If S is the image of \vec{r} under the map $(u, v) \mapsto \vec{r}(u, v)$, then $dS = |\vec{r}_u \times \vec{r}_v| \, dudv$.



DEFINITION. Given a surface $S = \vec{r}(R)$, where R is a domain in the plane and where $\vec{r}(u, v) = (x(u, v), y(u, v), z(u, v))$. The surface integral of $f(u, v)$ on S is defined as

$$\int \int_S f \, dS = \int \int_R f(u, v) |\vec{r}_u \times \vec{r}_v| \, dudv .$$

INTERPRETATION. If $f(x, y)$ measures a quantity then $\int \int_S f \, dS$ is the average of the function f on S .

EXPLANATION OF $|\vec{r}_u \times \vec{r}_v|$. The vector \vec{r}_u is a tangent vector to the curve $u \mapsto \vec{r}(u, v)$, when v is fixed and the vector \vec{r}_v is a tangent vector to the curve $v \mapsto \vec{r}(u, v)$, when u is fixed. The two vectors span a parallelogram with area $|\vec{r}_u \times \vec{r}_v|$. A little rectangle spanned by $[u, u + du]$ and $[v, v + dv]$ is mapped by \vec{r} to a parallelogram spanned by $[\vec{r}, \vec{r} + \vec{r}_u]$ and $[\vec{r}, \vec{r} + \vec{r}_v]$.

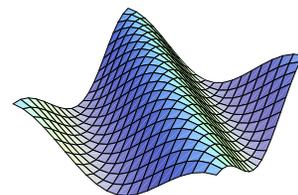
A simple case: consider $\vec{r}(u, v) = (2u, 3v, 0)$. This surface is part of the xy -plane. The parameter region R just gets stretched by a factor 2 in the x coordinate and by a factor 3 in the y coordinate. $\vec{r}_u \times \vec{r}_v = (0, 0, 6)$ and we see for example that the area of $\vec{r}(R)$ is 6 times the area of R .

POLAR COORDINATES. If we take $\vec{r}(u, v) = (u \cos(v), u \sin(v), 0)$, then the rectangle $[0, R] \times [0, 2\pi]$ is mapped into a flat surface which is a disc in the xy -plane. In this case $\vec{r}_u \times \vec{r}_v = (\cos(v), \sin(v), 0) \times (-u \sin(v), u \cos(v), 0) = (0, 0, u)$ and $|\vec{r}_u \times \vec{r}_v| = u = r$. We can explain the integration factor r in polar coordinates as a special case of a surface integral.

THE AREA OF THE SPHERE.

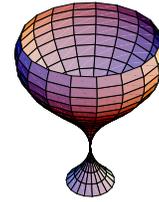
The map $\vec{r} : (u, v) \mapsto (L \cos(u) \sin(v), L \sin(u) \sin(v), L \cos(v))$ maps the rectangle $R : [0, 2\pi] \times [0, \pi]$ onto the sphere of radius L . We compute $\vec{r}_u \times \vec{r}_v = L \sin(v) \vec{r}(u, v)$. So, $|\vec{r}_u \times \vec{r}_v| = L^2 |\sin(v)|$ and $\int \int_R 1 \, dS = \int_0^{2\pi} \int_0^\pi L^2 \sin(v) \, dv du = 4\pi L^2$.

SURFACE AREA OF GRAPHS. For surfaces $(u, v) \mapsto (u, v, f(u, v))$, we have $\vec{r}_u = (1, 0, f_u(u, v))$ and $\vec{r}_v = (0, 1, f_v(u, v))$. The cross product $\vec{r}_u \times \vec{r}_v = (-f_u, -f_v, 1)$ has the length $\sqrt{1 + f_u^2 + f_v^2}$. The area of the surface above a region R is $\int \int_R \sqrt{1 + f_u^2 + f_v^2} \, dA$.



EXAMPLE. The surface area of the paraboloid $z = f(x, y) = x^2 + y^2$ is (use polar coordinates) $\int_0^{2\pi} \int_0^1 \sqrt{1 + 4r^2} r \, dr d\theta = 2\pi(2/3)(1 + 4r^2)^{3/2} / 8 \Big|_0^1 = \pi(5^{3/2} - 1)/6$.

AREA OF SURFACES OF REVOLUTION. If we rotate the graph of a function $f(x)$ on an interval $[a, b]$ around the x-axis, we get a surface parameterized by $(u, v) \mapsto \vec{r}(u, v) = (v, f(v) \cos(u), f(v) \sin(u))$ on $R = [0, 2\pi] \times [a, b]$ and is called a **surface of revolution**. We have $\vec{r}_u = (0, -f(v) \sin(u), f(v) \cos(u))$, $\vec{r}_v = (1, f'(v) \cos(u), f'(v) \sin(u))$ and $\vec{r}_u \times \vec{r}_v = (-f(v)f'(v), f(v) \cos(u), f(v) \sin(u)) = f(v)(-f'(v), \cos(u), \sin(u))$. The surface area is $\int \int |\vec{r}_u \times \vec{r}_v| \, dudv = 2\pi \int_a^b |f(v)| \sqrt{1 + f'(v)^2} \, dv$.



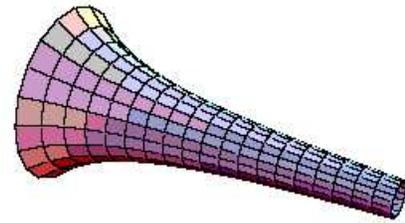
EXAMPLE. If $f(x) = x$ on $[0, 1]$, we get the surface area of a cone: $\int_0^{2\pi} \int_0^1 x \sqrt{1+1} \, dvdu = 2\pi\sqrt{2}/2 = \pi\sqrt{2}$.

P.S. In computer graphics, surfaces of revolutions are constructed from a few prescribed points $(x_i, f(x_i))$. The machine constructs a function (**spline**) and rotates

GABRIEL'S TRUMPET. Take $f(x) = 1/x$ on the interval $[1, \infty)$.

Volume: The volume is (use cylindrical coordinates in the x direction): $\int_1^\infty \pi f(x)^2 \, dx = \pi \int_1^\infty 1/x^2 \, dx = \pi$.

Area: The area is $\int_0^{2\pi} \int_1^\infty 1/x \sqrt{1+1/x^4} \, dx \geq 2\pi \int_1^\infty 1/x \, dx = 2\pi \log(x)|_1^\infty = \infty$.

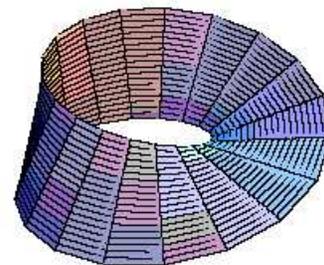


The Gabriel trumpet is a surface of finite volume but with infinite surface area! You can fill the trumpet with a finite amount of paint, but this paint does not suffice to cover the surface of the trumpet!

Question. How long does a Gabriel trumpet have to be so that its surface is 500cm^2 (area of sheet of paper)? Because $1 \leq \sqrt{1+1/x^4} \leq \sqrt{2}$, the area for a trumpet of length L is between $2\pi \int_1^L 1/x \, dx = 2\pi \log(L)$ and $\sqrt{2}2\pi \log(L)$. In our case, L is between $e^{500/(\sqrt{2}2\pi)} \sim 2 * 10^{24}\text{cm}$ and $e^{500/(2\pi)} \sim 4 * 10^{34}\text{cm}$. Note that the universe is about 10^{26} cm long (assuming that the universe expanded with speed of light since 15 Billion year). It could not accommodate a Gabriel trumpet with the surface area of a sheet of paper.

MÖBIUS STRIP. The surface $\vec{r}(u, v) = (2+v \cos(u/2) \cos(u), (2+v \cos(u/2)) \sin(u), v \sin(u/2))$ parametrized by $R = [0, 2\pi] \times [-1, 1]$ is called a **Möbius strip**.

The calculation of $|\vec{r}_u \times \vec{r}_v| = 4+3v^2/4+4v \cos(u/2)+v^2 \cos(u)/2$ is straightforward but a bit tedious. The integral over $[0, 2\pi] \times [-1, 1]$ is 17π .



QUESTION. If we build the Moebius strip from paper. What is the relation between the area of the surface and the weight of the surface?

REMARKS.

- 1) An OpenGL implementation of an Escher theme can be admired with "xlock -inwindow -mode moebius" on an X-terminal.
- 2) A patent was once assigned to the idea to use a Moebius strip as a **conveyor belt**. It would last twice as long as an ordinary one.

