Lecture 13: 7/23/2003, POLAR INTEGRATION — Maths21a, O. Knill

THINGS TO KEEP IN MIND.

- Double integrals can often be evaluated through iterated integrals

"Integrals have layers".

- $\int\int_R 1 \; dxdy$ is the **area** of the region R.

- $\int\int_R f(x,y) \; dxdy$ is the volume of the solid having the graph of f as the "roof" and the R in the $x-y$ plane as the floor.

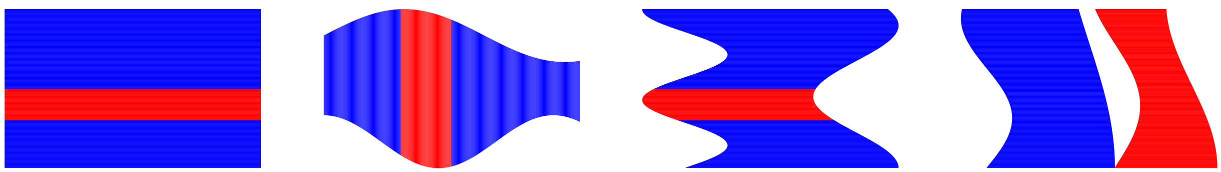
TYPES OF REGIONS.

$\int\int_R f \; dA = \int_a^b \int_c^c f(x,y) \; dydx$ **rectangle.**

$\int\int_R f \; dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) \; dydx$ **type I region.**

$\int\int_R f \; dA = \int_a^b \int_{h_1(y)}^{h_2(y)} f(x,y) \; dxdy$ **type II region.**

A general region, we try to cut it into pieces, where each piece is a Type I or Type II region.



Rectangle | Type I | Type II | To cut

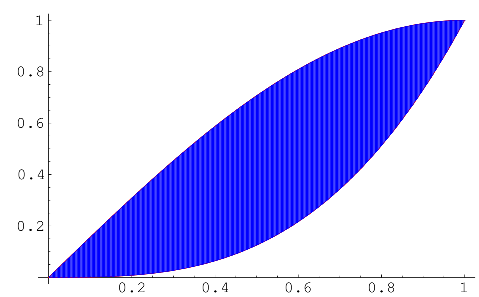
PROBLEM. Consider the type II integral

$$\int\int_R y dxdy = \int_0^1 \int_{2\arcsin(y)/\pi}^{y^{1/3}} y \; dxdy \; .$$

Draw the region R, write the integral as a type I integral and evaluate the later.

Solution. The curves $x = \frac{2\arcsin(y)}{\pi}$ and $x = y^{1/3}$ bounding the region are identical to the curves $y = \sin(\pi x/2)$ and $y = x^3$.

The region is bounded by these two curves:



The corresponding type I integral is $\int_0^1 \int_{x^3}^{\sin(\pi x/2)} y \; dydx = \int_0^1 \sin(\pi x/2)^2/2 - x^6/2 \; dx$.

POLAR COORDINATES. For many regions, it is better to use polar coordinates for integration:

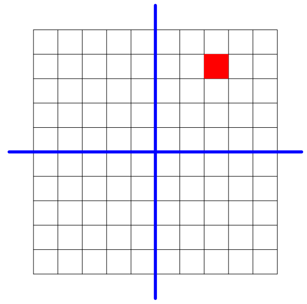
$$\boxed{\int\int_R f(x,y) \; dxdy = \int\int_R f(r,\theta) r \; drd\theta}$$

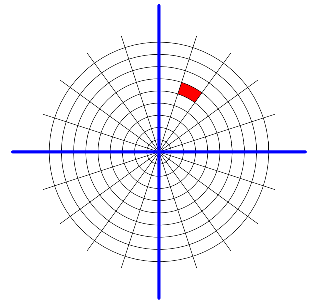
EXAMPLE. The area of the disc $\{x^2+y^2 \leq 1\}$ can be computed by treating the region as a type I region and doing the integral with $x = \sin(u), dx = \cos(u)du$: $\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1 \; dydx = \int_{-1}^{1} 2\sqrt{1-x^2} \; dx = \int_{-\pi/2}^{\pi/2} 2\cos^2(u) \; du = \pi$. It is easier to do that integral in polar coordinates:

$$\int_0^{2\pi} \int_0^1 r \; drd\theta = 2\pi r^2/2|_0^1 = \pi \; .$$

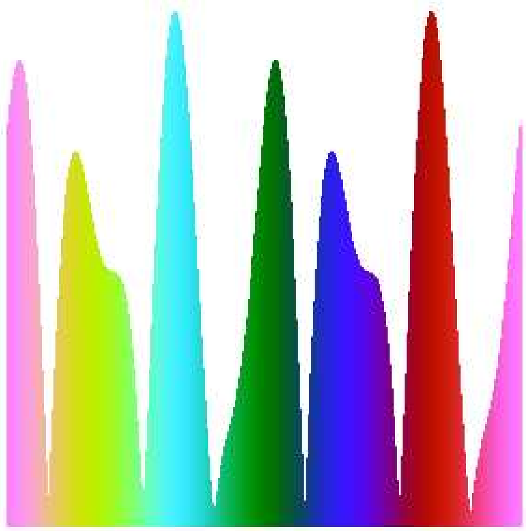
WHERE DOES THE FACTOR "r" COME FROM?

A small rectangle R with dimensions $d\theta dr$ in the (r,θ) plane is mapped by $T : (r,\theta) \mapsto (r\cos(\theta), r\sin(\theta))$ to a sector segment S in the (x,y) plane. It has approximately the area $rd\theta dr$. It is small for small r.

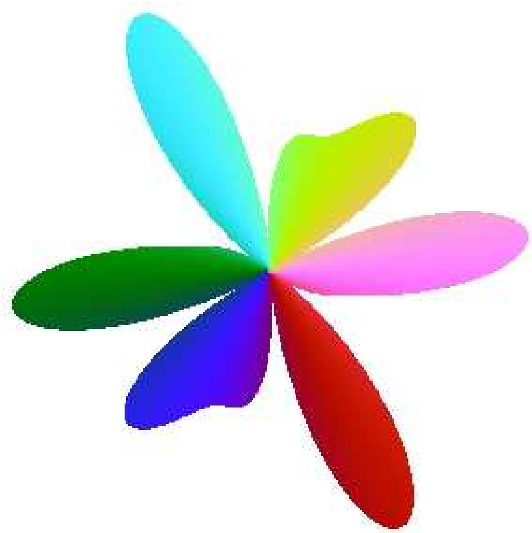




ROSES. We can now integrate over type I or type II regions in the (θ,r) plane. Examples are **roses**: $\{(\theta,r) \; | 0 \leq r \leq f(\theta)\}$ where $f(\theta)$ is a periodic function of θ.



The region R in the $\theta - r$ coordinates is a type I region



The region $S = T(R)$ in the $x-y$ coordinates is neither a type I nor a type II region.

EXAMPLE. Find the area of the region $\{(\theta, r(\theta)) \mid r(\theta) \leq \; | \cos(3\theta)| \; \}$.

$$\int\int_R y \; dxdy = \int_0^{2\pi} \int_0^{\cos(3\theta)} r \; dr \; d\theta = \int_0^{2\pi} \frac{\cos(3\theta)^2}{2} \; d\theta = \pi/2$$

EXAMPLE. Integrate $f(x,y) = y\sqrt{x^2+y^2}$ over the semi annulus $R = \{(x,y) \mid 1 < x^2+y^2 < 4, y > 0 \}$.

Solution.

$$\int_1^2 \int_0^{\pi} r\sin(\theta) r \; r \; d\theta dr = \int_1^2 r^3 \int_0^{\pi} \sin(\theta) \; d\theta dr = \frac{(2^4-1^4)}{4} \int_0^{\pi} \sin(\theta) \; d\theta = 15/2$$

For integration problems, where the region is part of an annulus, or if you see function with terms x^2+y^2 try to use polar coordinates $x = r\cos(\theta), y = r\sin(\theta)$.