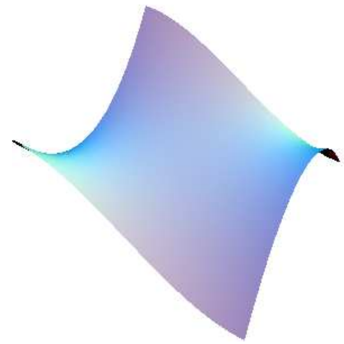
PDE's **Math 21a**

On this handout we deal with functions $f(t,x)$ of two variables. These functions satisfy equations containing partial derivatives called **partial differential equations** or shortly **PDE's**. The topic of PDE's would fill a course by itself. Finding and understanding solutions of such equations can be very difficult. The topic is used here only as an exercise for **partial differentiation**. No knowledge on PDE's will be required from you in this course. You should be able to verify however that a given function $f(t,x)$ satisfies a specific PDE.

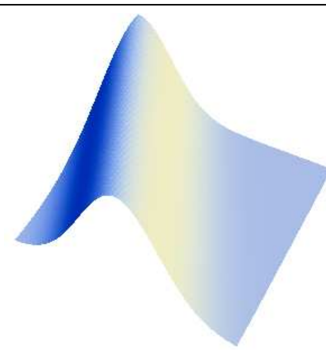
LAPLACE EQUATION. $\boxed{f_{xx} + f_{yy} = 0}$. A stationary temperature distribution on a plate satisfies this equation.

1) Verify that $\boxed{f(x,y) = x^3 - 3xy^2}$ satisfies the Laplace equation.



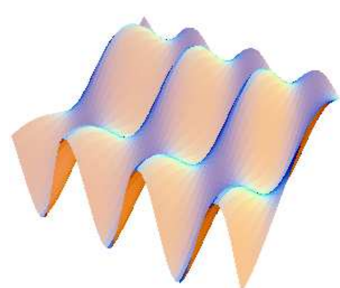
ADVECTION EQUATION. $\boxed{f_t = f_x}$. Models transport in a one-dimensional medium. It is also called a **transport equation**. In the homework, you look at a slightly more general case.

2) Verify that $\boxed{f(t,x) = e^{-(x+t)^2}}$ satisfy the advection equation $f_t(t,x) = f_x(t,x)$.



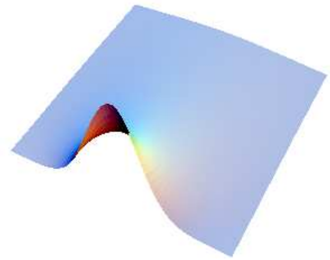
WAVE EQUATION. $\boxed{f_{tt} = f_{xx}}$. For fixed time t, the function $x \mapsto f(t,x)$ describes a string at that time.

3) Verify that $\boxed{f(t,x) = \sin(x-t) + \sin(x+t)}$ satisfies the wave equation $f_{tt}(t,x) = f_{xx}(t,x)$.



HEAT EQUATION. $\boxed{f_t = f_{xx}}$ For fixed time t, the function $x \mapsto f(t,x)$ is the temperature at the point x. The heat equation is also called **diffusion equation**.

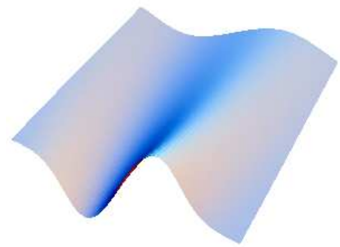
The function $\boxed{f(t,x) = \frac{1}{\sqrt{t}} e^{-x^2/(4t)}}$ satisfies the heat equation.



BURGER EQUATION. $\boxed{f_t + f f_x = f_{xx}}$ Describes one dimensional waves (i.e. at beach). In higher dimensions, it leads to the Navier Stokes equation. One of the millennium (10^6\$) problems is to solve the existence problem in 3D.

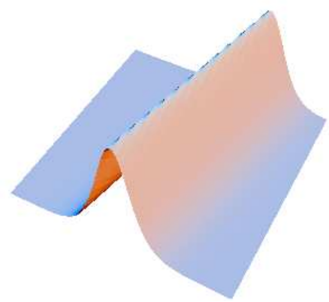
There are **N-wave** solutions $\boxed{f(t,x) = \frac{x}{t} \frac{\sqrt{\frac{1}{t}} e^{-x^2/(4t)}}{1+\sqrt{\frac{1}{t}} e^{-x^2/(4t)}}}$

Without f_{xx} term, solutions will break (form **shocks**).



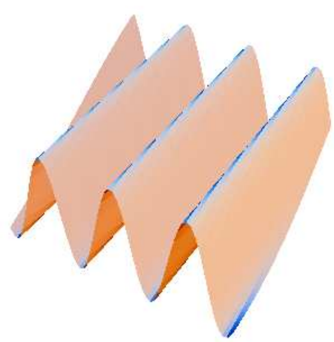
KDV-EQUATION. $\boxed{f_t + 6 f f_x + f_{xxx} = 0}$ Describes **water waves** in a narrow channel. First discovered by J. Scott Russel in 1838.

The solution $\boxed{f(t,x) = \frac{a^2}{2} \cosh^{-2}(\frac{a}{2}(x - a^2 t))}$ describes a wave with speed a^2 and amplitude $a^2/2$. It is called a **soliton**. Unlike linear waves, these **nonlinear waves** can travel with different speed: higher waves move faster. Solitons form a fancy research topic.



SCHRÖDINGER EQUATION. $\boxed{f_t = \frac{i\hbar}{2m} f_{xx}}$ Describes a free **quantum particle** of mass m.

A solution is $\boxed{f(t,x) = e^{i(kx - \frac{\hbar}{2m} k^2 t)}}$ models a particle with momentum $\hbar k$. The constant i satisfies $i^2 = -1$ (it is an imaginary number), $\hbar$ is the **Planck constant** $\hbar \sim 10^{-34} Js$.



"A great deal of my work is just **playing with equations** and seeing what they give. I don't suppose that applies so much to other physicists; I think it's a peculiarity of myself that I like to play about with equations, just **looking for beautiful mathematical relations** which maybe don't have any physical meaning at all. Sometimes they do." - Paul A. M. Dirac.

Dirac discovered a PDE describing the electron which is consistent both with quantum theory and special relativity. This won him the Nobel Prize in 1933. Dirac's equation could have two solutions, one for an electron with positive energy, and one for an electron with negative energy. Dirac interpretated the later as an **antiparticle**: the existence of antiparticles was later confirmed.