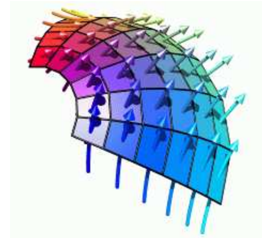
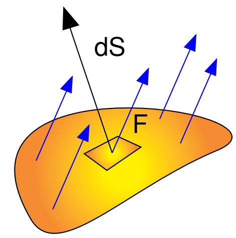
FLUX INTEGRAL OF A VECTOR FIELD THROUGH SURFACE. If S is a surface, and F is a vector field in space we define an integral $\int\int_S F \cdot dS$ which is called the **flux** of F through the surface S.



DEFINITION. Given a surface $S = r(R)$, where R is a domain in the plane and where $r(u, v) = (x(u, v), y(u, v), z(u, v))$ is the parameterization of the surface. The **flux integral** of F through S is defined as the double integral

$$\boxed{\int\int_S F \cdot dS = \int\int_R F(r(u, v)) \cdot (r_u \times r_v)\, dudv}$$

The notation $dS = (r_u \times r_v)\, dudv$ means an infinitesimal vector in the normal direction at the surface.



INTERPRETATION. If $F =$ fluid velocity field, then $\int\int_S F \cdot dS$ is the flux of fluid passing through S.

EXAMPLE. Let $F(x, y, z) = (0, 1, z^2)$ and let S be the sphere with

$$r(u, v) = (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$$

, $r_u \times r_v = \sin(v) r$ so that $F(r(u, v)) = (0, 1, \cos^2(v))$ and

$$\int_0^{2\pi}\int_0^{\pi} \sin(v)(0, 1, \cos^2(v)) \cdot (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))\, dudv\ .$$

The integral is $\int_0^{2\pi}\int_0^{\pi} \sin^2(v)\sin(u) + \cos^3(v)\sin(v)\, dudv = 0$.

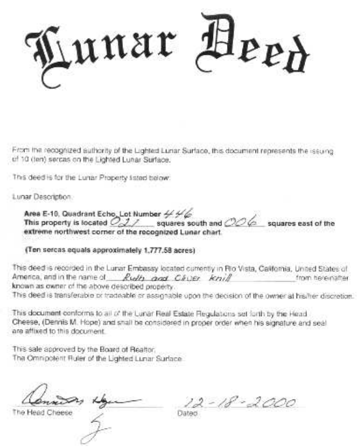
Look at the vector field. Most flux passes in at the south pole, most flux passes out at the north pole. However, there is some symmetry and what enters in the south leaves in the north. This explains why the total flux is zero.

WHAT IS THE FLUX INTEGRAL? Because $n = r_u \times r_v/|r_u \times r_v|$ is a unit vector normal to the surface and on the surface, $F \cdot n$ is the normal component of the vector field with respect to the surface, we can write $\int\int_S F \cdot dS = \int\int F \cdot n\, dS$. The function $F \cdot n$ is the scalar projection of of F in the normal direction. Whereas the formula $\int\int 1\, dS$ gave the area of the surface with $dS = |r_u \times r_v| dudv$, the flux integral weights each area element dS with the normal component of the vector field $f(u, v) = (F(r(u, v)) \cdot n(r(u, v))$.

BIKING IN THE RAIN.
I experience (like last Monday) that during rain one gets soaked more by the rain when biking fast then by biking slow or walking foot. Why? For a biker who drives with speed V along the x axis, the rain is a fluid with velocity $F = (-V, 0, W)$, where W is the speed of the rain drops falling along the z-axis. If the biker is a rectangular box $[0, 1/2]x[0, 1]x[1/4, 1/3]$ we can calculate the flux through that surface. Parameterize the front by $r(u, v) = (0, u, v)$ in the yz-plane. The water soaked up in unit time by the front clothes is the flux of the rain through that surface. We know $r_u \times r_v = (1, 0, 0)$ and $\int\int_S F \cdot dS = \int_0^{1/2}\int_0^1 (-V, 0, W) \cdot (1, 0, 0)\, dudv = -V/2$. The flux through the side surface is zero (if the rain falls vertically) and the flux through the top is constant, namely $W/12$. The total flux is $W/12 - V/2$. You catch more rain if you drive faster.

PROPERTY ON THE MOON. More than two years ago, in lack of a better present idea, I bought property on the moon. (For 30 Us-Dollars one could get 1'777.58 Acres, the size of Manhattan)) at the Lunar Embassy (www.moonshop.com). An acre is now sold for 15 Dollars.

The coordinates of the parcel are $34E, 26N$. The moon has a radius of r=1737 km. The surface is $3.8 10^6 km^2$. Multiply this by 247 gives about 10 billion (10^{10}) acres (more than an acre for each person on the world). The moon surface is parameterized by $r(u,v) = (r\cos(u)\sin(v), r\sin(u)\sin(v), r\cos(v)$ where u is the latitude and $\pi/2 - v$ is the latitude. If we look at the moon from the direction $(1,0,0)$, how big does the area of a real estate $S = r([a,b] \times [c,d])$ appear?



Lunar Deed

From the recognized authority of the Lighted Lunar Surface, this document represents the issuing of 10 (ten) sercas on the Lighted Lunar Surface.

This deed is for the Lunar Property listed below:

Lunar Description:

Area E-10, Quadrant Echo, Lot Number 446
This property is located 021 **squares south and** 006 **squares east of the extreme northwest corner of the recognized Lunar chart.**

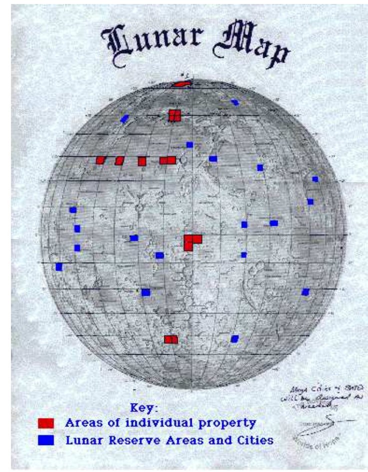
(Ten sercas equals approximately 1,777.58 acres)

This deed is recorded in the Lunar Embassy located currently in Rio Vista, California, United States of America, and in the name of Ruth and Oliver Knill from hereinafter known as owner of the above described property.
This deed is transferable or tradeable or assignable upon the decision of the owner at his/her discretion.

This document conforms to all of the Lunar Real Estate Regulations set forth by the Head Cheese, (Dennis M. Hope) and shall be considered in proper order when his signature and seal are affixed to this document.

This sale approved by the Board of Realtor,
The Omnipotent Ruler of the Lighted Lunar Surface

The Head Cheese — Dated 12-18-2000



The flux of the vector field $(1,0,0)$ through the surface is $\int\int_S F \cdot dS = \int_a^b \int_c^d r^2 \cos(u)\sin^2(v)\,dvdu$. This is also the area we **see**. You notice that the visible area is small for $\cos(u)$ small or $\sin(v)$ small, which is near $90E, 90W$ or near the poles.

EXAMPLE. Calculate the flux of the vector field $F(x,y,z) = (1,2,3z)$ through the paraboloid $z = x^2 + y^2$ lying above the region $x^2 + y^2 \leq 1$.

Parameterize it using polar coordinates $r(u,v) = (u\cos(v), u\sin(v), u^2)$ where $r_u \times r_v = (-2u^2\cos(v), -2u^2\sin(v), u)$ and $F(r(u,v)) = (1,2,3u^2)$. We get $\int_S F \cdot dS = \int_0^{2\pi}\int_0^1 (-2u^2\cos(v) - 4u^2\sin(v) + 3u^3)\,dudv = 1$.

WHERE CAN FLUX INTEGRALS APPEAR?

- **Fluid dynamics** The flux of the velocity vector field of a membrane through a surface is the volume of the fluid which passes through that surface in unit time. Often one looks at incompressible fluids which means $\text{div}(v) = 0$. In this case, the amount of fluid which passes through a closed surface like a sphere is zero because what enters also has toe leave the surface.
- **Electromagnetism** The change of the flux of the magnetic field through a surface is related to the voltage at the boundary of the surface. We will see why one of the Maxwell equations $\text{div}(B) = 0$ means that there are no **magnetic monopoles** and see how divergence is related to flux.
- **Magnetohydrodynamics** Magnetohydrodynamics deals with electrically charged "fluids" like plasmas in the sun or solar winds etc. The velocity field of the wind induces a current and the flux of the field trough a surface can be used to calculate the magnetic field.
- **Gravity** We will see that the flux of a field through a closed surface like the sphere is related to the amount of source inside the the surface. In gravity, where F is the gravitational field, the flux of the gravitational field through a closed surface is proportional to the mass inside the surface. Mass is a "source for gravitational field".
- **Thermodynamics**. The heat flux through a surface like the walls of a house are related to the amount of heat which is produced inside the house. A badly isolated house needs has a large heat flux and needs a large source of heating.
- **Particle physics**. When studying the collision of particles, one is interested in the relation between the incoming flux and out-coming fluxes depending on the direction. The key word is: "differential cross-section".
- **Optics**. Measurements of "light fluxes" are important for lasers, astronomy, photography.