

CRITICAL POINTS. A point (x_0, y_0) is called a **critical point** of $f(x, y)$ if $\nabla f(x_0, y_0) = (0, 0)$. A point (x_0, y_0, z_0) is called a **critical point** of $f(x, y, z)$ if $\nabla f(x_0, y_0, z_0) = (0, 0, 0)$. Critical points are also called **stationary points**.

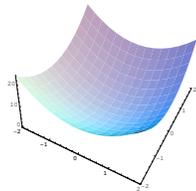
EXAMPLE 1. $f(x, y) = \sin(x^2 + y) + y$. The gradient is $\nabla f(x, y) = (2x \cos(x^2 + y), \cos(x^2 + y) + 1)$. For a critical points, we must have $x = 0$ and $\cos(y) + 1 = 0$ which means $\pi + k2\pi$. The critical points are at $(0, \pi), (0, 3\pi), \dots$

EXAMPLE 2. ("volcano") $f(x, y) = (x^2 + y^2)e^{-x^2 - y^2}$. The gradient $\nabla F = (2x - 2(x^2 + y^2), 2y - 2(x^2 + y^2))e^{-x^2 - y^2}$ vanishes at $(0, 0)$ and on the circle $x^2 + y^2 = 1$. There are ∞ many critical points.

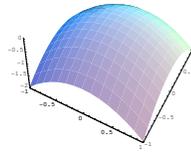
EXAMPLE 3 ("pendulum") $f(x, y) = -g \cos(x) + y^2/2$ is the energy of the pendulum. The gradient $\nabla F = (y, -g \sin(x))$ is $(0, 0)$ for $x = 0, \pi, 2\pi, \dots, y = 0$. These points are equilibrium points, where the pendulum is at rest.

EXAMPLE 4 ("Volterra Lodka") $f(x, y) = a \log(y) - by + c \log(x) - dx$. (This function is left invariant by the flow of the Volterra Lodka differential equation $\dot{x} = ax - bxy, \dot{y} = -cy + dxy$ which you might have seen in Math1b.) The point $(c/d, a/b)$ is a critical point.

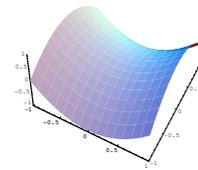
TYPICAL EXAMPLES.



$$f(x, y) = x^2 + y^2$$

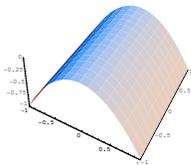


$$f(x, y) = -x^2 - y^2$$

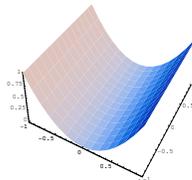


$$f(x, y) = x^2 - y^2$$

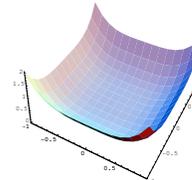
EXAMPLES WITH DISCRIMINANT $D = \det(H) = 0$.



$$f(x, y) = x^2$$



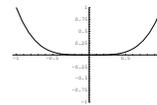
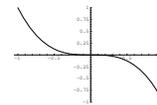
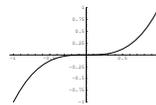
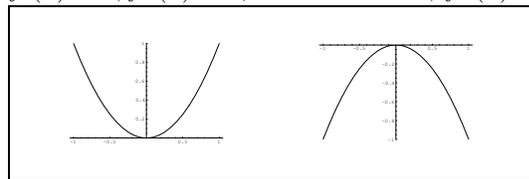
$$f(x, y) = -x^2$$



$$f(x, y) = x^4 + y^4$$

CLASSIFICATION OF CRITICAL POINTS IN 1 DIMENSION.

$f'(x) = 0, f''(x) > 0$, local minimum, $f''(x) < 0$ local maximum, $f'' = 0$ undetermined.



CLASSIFICATION OF CRITICAL POINTS FOR FUNCTIONS OF TWO VARIABLES.

(Second derivative test). Let $f(x, y)$ be a function of two variables with a critical point (x_0, y_0) . Define $D = f_{xx}f_{yy} - f_{xy}^2$, the discriminant. (Remark: With the **Hessian matrix** $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$ we can write $D = \det(H)$ as a **determinant** (see Math21b).)

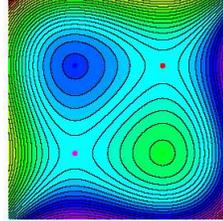
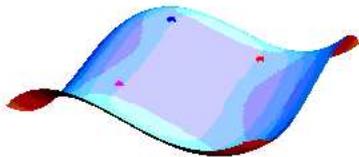
If $D > 0$ and $H_{11} > 0 \Rightarrow$ local minimum (bottom of valley)
 If $D > 0$ and $H_{11} < 0 \Rightarrow$ local maximum (top of mountain).
 If $D < 0 \Rightarrow$ saddle point (mountain pass).

In the case $D = 0$, we would need higher derivatives to determine the nature of the the critical point.

EXAMPLE. (A "napkin").

The function $f(x, y) = x^3/3 - x - (y^3/3 - y)$ has the gradient $\nabla f(x, y) = (x^2 - 1, -y^2 + 1)$. It is the zero vector at the 4 critical points $(1, 1), (-1, 1), (1, -1)$ and $(-1, -1)$. The Hessian matrix is $H = f''(x, y) = \begin{bmatrix} 2x & 0 \\ 0 & -2y \end{bmatrix}$.

$H(1, 1) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$	$H(-1, 1) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$	$H(1, -1) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$	$H(-1, -1) = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$
$D = -4$ Saddle point	$D = 4, f_{xx} = -2$ Local maximum	$D = 4, f_{xx} = 2$ Local minimum	$D = -4$ Saddle point



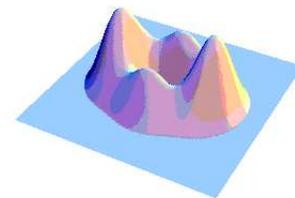
MAXIMA-MINIMA. To determine the maximum or minimum of $f(x, y)$ on a domain, determine all critical points **inside the domain**, and compare their values with maxima or minima **at the boundary**.

EXAMPLE 5. Find all the critical points of $f(x, y) = 2x^2 - x^3 - y^2$. We have $\nabla f(x, y) = (4x - 3x^2, -2y)$. The critical points are at $(4/3, 0)$ and $(0, 0)$. The Hessian is $H(x, y) = \begin{pmatrix} 4 - 6x & 0 \\ 0 & -2 \end{pmatrix}$. At $(0, 0)$, the discriminant is -8 so that this is a saddle point. At $(4/3, 0)$, the discriminant is 8 and $H_{11} = 4/3$, so that $(4/3, 0)$ is a local maximum.

WHY DO WE CARE ABOUT CRITICAL POINTS?

- Critical points are candidates for extrema like maxima or minima.
- Knowing all the critical points and their nature tells a lot about the function.
- Critical points are physically relevant. Examples are configurations with lowest energy).

A CURIOUS OBSERVATION: (The island theorem) Let $f(x, y)$ be the height on an island. Assume there are only finitely many critical points on the island and all of them have nonzero determinant. Label each critical point with a $+1$ "charge" if it is a maximum or minimum, and with -1 "charge" if it is a saddle point. Sum up all the charges and you will get 1 , independent of the function. This property is an example of an "index theorem", a prototype for important theorems in physics and mathematics.



CRITICAL POINTS IN PHYSICS. (informal) Most physical laws are based on the principle that the equations are critical points of a functional (in general in infinite dimensions).

• **Newton equations.** (Classical mechanics) A particle of mass m moving in a field V along a path $\gamma : t \mapsto r(t)$ extremizes the integral $S(\gamma) = \int_a^b m r'(t)^2/2 - V(r(t)) dt$. Critical points γ satisfy the Newton equations $m r''(t)/2 - \nabla V(r(t)) = 0$.

• **Maxwell equations.** (Electromagnetism) The electromagnetic field (E, B) extremizes the Integral $S(E, B) = \frac{1}{8\pi} \int (E^2 - B^2) dV$ over space time. Critical points are described by the Maxwell equations in vacuum.

• **Einstein equations** (General relativity) If g is a dot product which depends on space and time, and R is the "curvature" of the corresponding curved space time, then $S(g) = \int \int_R dV(g)$ is a function of g for which critical points g satisfy the Einstein equations in general relativity.

OTHER WAYS TO FIND CRITICAL POINTS. Some ideas: walk in the direction of the gradient until you reach a local maximum or walk backwards to reach a local minima. To find saddle points, consider the shortest path connecting two local minima and take the maximum along this path.