Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1)	(20 points)

T	F (highlighted)

For any two nonzero vectors $\vec{v}, \vec{w}$ the vector $((\vec{v} \times \vec{w}) \times \vec{v}) \times \vec{v})$ is parallel to $\vec{w}$.

Solution:
Take $v = (1,0,0), w = (0,1,0)$ so that $\vec{v} \times \vec{w} = (0,0,1)$ and $(\vec{v} \times \vec{w}) \times \vec{v}) = (0,1,0)$ and $((\vec{v} \times \vec{w}) \times \vec{v}) \times \vec{v}) = (0,0,1)$.

T	F (highlighted)

The cross product satisfies the law $(\vec{u} \times \vec{v}) \times \vec{w} = \vec{u} \times (\vec{v} \times \vec{w})$.

Solution:
Take $\vec{v} = \vec{w}$, then the right hand side is the zero vector while the left hand side is not zero in general (for example if $u = i, v = j$).

T (highlighted)	F

If the curvature of a smooth curve $\vec{r}(t)$ in space is defined and zero for all t, then the curve is part of a line.

Solution:
One can see that with the formula $\kappa(t) = |r'(t) \times r''(t)|/|r'(t)|^3$ which shows that the acceleration $r''(t)$ is in the velocity direction at all times. One can also see it intuitively or with the definition $\kappa(t) = |T'(t)|/|r'(t)|$. If curve is not part of a line, then T has to change which means that κ is not zero somewhere.

T (highlighted)	F

The curve $\vec{r}(t) = (1-t)A + tB, t \in [0,1]$ connects the point A with the point B.

Solution:
The curve is a parameterization of a line and for $t = 0$, one has $\vec{r}(0) = A$ and for $t = 1$ one has $\vec{r}(1) = B$.

T (highlighted)	F

For every c, the function $u(x,t) = (2\cos(ct) + 3\sin(ct))\sin(x)$ is a solution to the wave equation $u_{tt} = c^2 u_{xx}$.

Solution:
Just differentiate.

T (highlighted)	F

The length of the curve $\vec{r}(t) = (t, \sin(t))$, where $t \in [0, 2\pi]$ is $\int_0^{2\pi} \sqrt{1 + \cos^2(t)}\, dt$.

Solution:
The speed at time t is $|\vec{r}'(t)| = \sqrt{1+\cos^2(t)}$.

T	**F**

Let (x_0, y_0) be the maximum of $f(x,y)$ under the constraint $g(x,y) = 1$. Then $f_{xx}(x_0, y_0) < 0$.

Solution:
While this would be true for $g(x,y) = f(y)$, where the constraint is a straight line parallel to the y axes, it is false in general.

T	**F**

The function $f(x,y,z) = x^2 - y^2 - z^2$ decreases in the direction $(2,-2,-2)/\sqrt{8}$ at the point $(1,1,1)$.

Solution:
It **increases** in that direction.

T	F

Assume $\vec{F}$ is a vector field satisfying $|\vec{F}(x,y,z)| \leq 1$ everywhere. For every curve $C : \vec{r}(t)$ with $t \in [0,1]$, the line integral $\int_C \vec{F} \cdot \vec{dr}$ is less or equal than the arc length of C.

Solution:
$|\vec{F} \cdot \vec{r}'| \leq |\vec{F}||\vec{r}'| \leq |\vec{r}'|$.

T	F

Let $\vec{F}$ be a vector field which coincides with the unit normal vector $\vec{N}$ for each point on a curve C. Then $\int_C \vec{F} \cdot \vec{dr} = 0$.

Solution:
The vector field is orthogonal to the tangent vector to the curve.

T	**F**

The divergence of the gradient of any $f(x,y,z)$ is always zero.

Solution:
$\text{div}(\text{grad}(f)) = \Delta f$ is the Laplacian of f.

T | F — selected: **T**

For every function f, one has $\text{div}(\text{curl}(\text{grad}(f))) = 0$.

Solution:
Both because $\text{div}(\text{curl}(F) = 0$ and $\text{curl}(\text{grad}(f) = 0$.

T | F — selected: **F**

If for two vector fields $\vec{F}$ and $\vec{G}$ one has $\text{curl}(\vec{F}) = \text{curl}(\vec{G})$, then $\vec{F} = \vec{G} + (a, b, c)$, where a, b, c are constants.

Solution:
One can also have $\vec{F} = \vec{G} + \text{grad}(f)$ which are vectorfields with the same curl.

T | F — selected: **F**

For every vector field $\vec{F}$ the identity $\text{grad}(\text{div}(\vec{F})) = \vec{0}$ holds.

Solution:
$F = (x^2, y^2, z^2)$ has $\text{div}(F) = (2x, 2y, 2z)$ which has a nonzero gradient.

T | F — selected: **T**

If a nonempty quadric surface $g(x, y, z) = ax^2 + by^2 + cz^2 = 5$ can be contained inside a finite box, then $a, b, c \geq 0$.

Solution:
If one or two of the constants a, b, c are negative, we have a hyperboloid which all can not be contained into a finite space. If all three are negative, then the surface is empty.

T | F — selected: **F**

If $\vec{F}$ is a vector field in space then the flux of $\vec{F}$ through any closed surface S is 0.

Solution:
While it is true that the flux of $\text{curl}(F)$ vanishes through every closed surface, this is not true for $\vec{F}$ itself. Take for example $F = (x, y, z)$.

T | F — selected: **F**

If $\text{div}(\vec{F})(x, y, z) = 0$ for all (x, y, z), then $\text{curl}(\vec{F}) = (0, 0, 0)$ for all (x, y, z).

Solution:
Take $(-y, x, 0)$ for example.

T	F

The flux of the vector field $\vec{F}(x,y,z) = (y+z, y, -z)$ through the boundary of a solid region E is equal to the volume of E.

Solution:
By the divergence theorem, the flux through the boundary is $\int\int\int_E \operatorname{div}(F)\ dV$ but $\operatorname{div}(F) = 0$. So the flux is zero.

T	F

If in spherical coordinates the equation $\phi = \alpha$ (with a constant α) defines a plane, then $\alpha = \pi/2$.

Solution:
Otherwise, it is would be a cone (or for $\alpha = 0$ or $\alpha = \pi$ a half line).

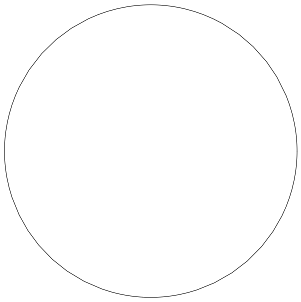
T	F

For every function $f(x,y,z)$, there exists a vector field $\vec{F}$ such that $\operatorname{div}(\vec{F}) = f$.

Solution:
In order to solve $P_x + Q_y + R_z = f$ just take $F = (0, 0, \int_0^z f(x,y,w)\ dw)$.

Problem 2) (10 points)

Match the equations with the objects. No justifications are needed.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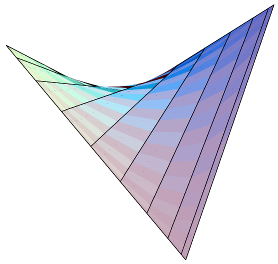
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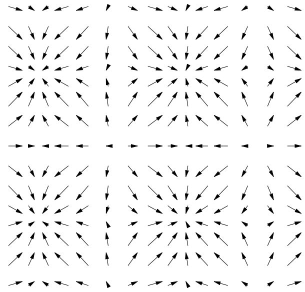
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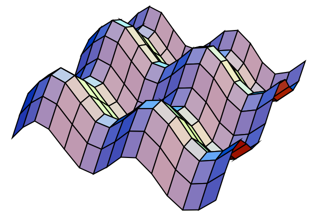
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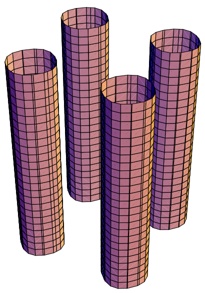
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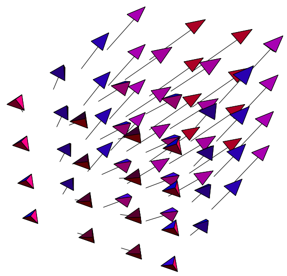
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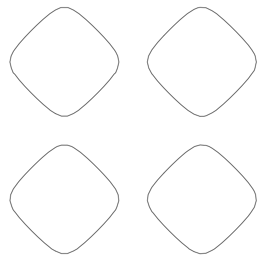
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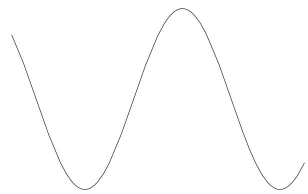

VIII

Enter I,II,III,IV,V,VI,VII,VIII here	Equation
	$g(x,y,z) = \cos(x) + \sin(y) = 1$
	$y = \cos(x) - \sin(x)$
	$\vec{r}(t) = (\cos(t), \sin(t))$
	$\vec{r}(u,v) = (\cos(u), \sin(v), \cos(u)\sin(v))$
	$\vec{F}(x,y,z) = (\cos(x), \sin(x), 1)$
	$z = f(x,y) = \cos(x) + \sin(y)$
	$g(x,y) = \cos(x) - \sin(y) = 1$
	$\vec{F}(x,y) = (\cos(x), \sin(x))$

Solution:

Enter I,II,III,IV,V,VI,VII,VIII here	Equation
V	$g(x, y, z) = \cos(x) + \sin(y) = 1$
VIII	$y = \cos(x) - \sin(x)$
I	$\vec{r}(t) = (\cos(t), \sin(t))$
II	$\vec{r}(u, v) = (\cos(u), \sin(v), \cos(u)\sin(v))$
VI	$\vec{F}(x, y, z) = (\cos(x), \sin(x), 1)$
IV	$z = f(x, y) = \cos(x) + \sin(y)$
VII	$g(x, y) = \cos(x) - \sin(y) = 1$
III	$\vec{F}(x, y) = (\cos(x), \sin(x))$

Problem 3) (10 points)

Mark with a cross in the column below "conservative" if a vector fields is conservative (that is if $\text{curl}(\vec{F})(x, y, z) = (0, 0, 0)$ for all points (x, y, z)). Similarly, mark the fields which are incompressible (that is if $\text{div}(\vec{F})(x, y, z) = 0$ for all (x, y, z)). No justifications are needed.

Vectorfield	conservative $\text{curl}(\vec{F}) = \vec{0}$	incompressible $\text{div}(\vec{F}) = 0$
$\vec{F}(x, y, z) = (-5, 5, 3)$		
$\vec{F}(x, y, z) = (x, y, z)$		
$\vec{F}(x, y, z) = (-y, x, z)$		
$\vec{F}(x, y, z) = (x^2 + y^2, xyz, x - y + z)$		
$\vec{F}(x, y, z) = (x - 2yz, y - 2zx, z - 2xy)$		

Solution:

Vectorfield	conservative $\text{curl}(\vec{F}) = \vec{0}$	incompressible $\text{div}(\vec{F}) = 0$
$\vec{F}(x,y,z) = (-5,5,3)$	X	X
$\vec{F}(x,y,z) = (x,y,z)$	X	
$\vec{F}(x,y,z) = (-y,x,z)$		
$\vec{F}(x,y,z) = (x^2+y^2, xyz, x-y+z)$		
$\vec{F}(x,y,z) = (x-2yz, y-2zx, z-2xy)$	X	

Problem 4) (10 points)

Let E be a parallelogram in three dimensional space defined by two vectors $\vec{u}$ and $\vec{v}$.

a) (3 points) Express the diagonals of the parallelogram as vectors in terms of $\vec{u}$ and $\vec{v}$.

b) (3 points) What is the relation between the length of the crossproduct of the diagonals and the area of the parallelogram?

c) (4 points) Assume that the diagonals are perpendicular. What is the relation between the lengths of the sides of the parallelogram?

Solution:

a) first diagonal $\boxed{u+v}$, second diagonal $\boxed{u-v}$.

b) $(u+v) \times (u-v) = 2v \times u = \boxed{\text{2 times area of of parallelogram}}$.

c) $(u+v) \cdot (u-v) = |u|^2 - |v|^2 = 0$, so that $\boxed{|u| = |v|}$.

Problem 5) (10 points)

Find the volume of the largest rectangular box with sides parallel to the coordinate planes that can be inscribed in the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} = 1$.

Solution:
The volume of the box is $8xyz$. The Lagrange equations are

$$\begin{aligned} 8yz &= \lambda x/2 \\ 8xz &= \lambda 2y/9 \\ 8xy &= \lambda 2z/25 \end{aligned}$$

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} - 1 = 0$$

We can solve this by solving the first three equations for λ and expressing y, z by x, plugging this into the fourth equation. An other way to solve this is to multiply the first equation with x, the second with y and third with z.
The solution is $\boxed{x = 2/\sqrt{3}, y = \sqrt{3}, z = 5/\sqrt{3}}$. The maximal volume is $8xyz = 80/\sqrt{3}$.

Problem 6) (10 points)

Evaluate

$$\int_0^8 \int_{y^{1/3}}^2 \frac{y^2 e^{x^2}}{x^8} \, dxdy .$$

Solution:
This type II integral can not be computed as it is. We write it as a type I integral: from the boundary relation $x = y^{1/3}$ we obtain $y = x^3$ and $y = 8$ corresponds to $x = 2$:

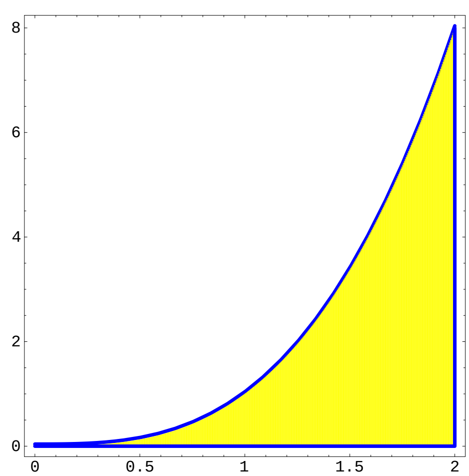
$$\int_0^2 \int_0^{x^3} \frac{y^2 e^{x^2}}{x^8} \, dydx$$

$$\int_0^2 \frac{x^9}{3} \frac{e^{x^2}}{x^8} \, dx$$

$$\int_0^2 \frac{x}{3} e^{x^2} \, dx$$

$$e^{x^2}/6|_0^2 = (e^4 - 1)/6$$

The result is $\boxed{\frac{e^4-1}{6}}$.



Problem 7) (10 points)

Evaluate $\int\int_D \frac{2xy}{x^2+y^2}\, dxdy$, where D is the intersection of the annulus $1 \leq x^2 + y^2 \leq 2$ with the second quadrant $\{x \leq 0, y \geq 0\}$.

Solution:
Use polar coordinates $x = r\cos(\theta), y = r\sin(\theta)$ for which $2xy = r^2\sin(2\theta)$ and $x^2+y^2 = r^2$. The integral is $\int_1^{\sqrt{2}} \int_{\pi/2}^{\pi} r^2 \sin(2\theta)/r^2 \; r d\theta dr = -(r^2)/2|_1^{\sqrt{2}} \cos(2\theta)/2|_{\pi/2}^{\pi} = 1/2$.

Problem 8) (10 points)

a) (3 points) Find all the critical points of the function $f(x, y) = -(x^4 - 8x^2 + y^2 + 1)$.

b) (3 points) Classify the critical points.

c) (2 points) Locate the local and absolute maxima of f.

d) (2 points) Find the equation for the tangent plane to the graph of f at each absolute maximum.

Solution:
a) $\boxed{(\pm 2, 0) \text{ and } (0, 0)}$.
b) $\boxed{(-2, 0) \text{ is a local maximum with value } 15}$.
$\boxed{(0, 0) \text{ is a saddle with value } -1}$.
$\boxed{(2, 0) \text{ is a maximum with value } 15}$.
c) $\boxed{\text{The local maxima are } (\pm 2, 0)}$. They are also the absolute maxima because f decays at infinity.
d) To calculate the tangent plane at the maximum, write the graph of f as a level surface $g(x, y, z) = z - f(x, y)$. The gradient of g is orthogonal to the surface. We have $\nabla g = (0, 0, 1)$ so that the tangent plane has the equation $z = d = const$. Plugging in the point $(\pm 2, 0, 15)$ shows that $\boxed{z = 15}$ is the equation for the tangent plane for both maxima.

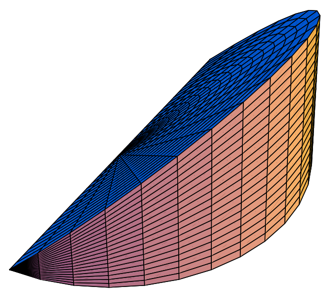
Problem 9) (10 points)

Find the volume of the wedge shaped solid that lies above the xy-plane and below the plane $z = x$ and within the cylinder $x^2 + y^2 = 4$.

Solution:
Use polar coordinates and note that the wedge is above the right side of the unit disc:

$$\int_0^2 \int_{-\pi/2}^{\pi/2} r^2 \cos(\theta)\, d\theta dr = 16/3$$

The solution is $\boxed{16/3}$.



Problem 10) (10 points)

Let the curve C be parametrized by $\vec{r}(t) = (t, \sin t, t^2 \cos t)$ for $0 \leq t \leq \pi$. Let $f(x, y, z) = z^2 e^{x+2y} + x^2$ and $\vec{F} = \nabla f$. Find $\int_C \vec{F} \cdot d\vec{r}$.

Solution:
Use the fundamental theorem of line integrals. The result is $f(r(\pi)) - f(r(0)) = f(\pi, 0, -\pi^2) - f(0, 0, 0) = \pi^4 e^\pi + \pi^2 - 0 = \boxed{\pi^4 e^\pi + \pi^2}$.

Problem 11) (10 points)

a) Find the linear approximation $L(x, y)$ of $f(x, y) = \sqrt{4 + 2x^2 + 4y^2}$ at the point $(x, y) = (2, 1)$.
b) Find the equation for the tangent line to the level curve of $f(u, v)$ at $(2, 1)$.

Solution:
a) $L(x, y) = 4 + (x - 2) + 2(y - 1) = x + 2y$.
b) $x + 2y = 4$.

Problem 12) (10 points)

Evaluate the line integral of the vector field $\vec{F}(x, y) = (y^2, x^2)$ in the clockwise direction around the triangle in the xy-plane defined by the points $(0,0),(1,0)$ and $(1,1)$ in two ways:

a) (5 points) by evaluating the three line integrals.
b) (5 points) using Greens theorem.

Solution:
The problem asks to do this in the clockwise direction. We do it in the counterclockwise direction and change then the sign.
a) $\int_0^1 F(t,0)\cdot(1,0)\ dt+\int_0^1 F(1,t)\cdot(0,1)\ dt+\int_0^1 F(1-t,1-t)\cdot(-1,-1)\ dt = 0+1-2/3 = 1/3$.
So, the result for the clockwise direction is $\boxed{-1/3}$.
b) The curl of F is $2x-2y$.

$$\int_0^1 \int_0^x (2x-2y)dy\ dx = \int_0^1 2x^2 - x^2\ dx = 1/3$$

So, the result for the clockwise direction is $\boxed{-1/3}$.

Problem 13)	(10 points)

Use Stokes theorem to evaluate the line integral of $\vec{F}(x,y,z) = (-y^3, x^3, -z^3)$ along the curve $\vec{r}(t) = (\cos(t), \sin(t), 1-\cos(t)-\sin(t))$ with $t \in [0, 2\pi]$.

Solution:
The curve is contained in the graph of the function $f(x,y) = 1-x-y$ which is parameterized by $r(u,v) = (u,v,1-u-v)$ and has the normal vector $r_u \times r_v = (1,0,-1)\times(0,1,-1) = (1,1,1)$. The curl of F is $(0,0,3x^2+3y^2)$ so that $F(r(u,v))\cdot(r_u\times r_v) = 3(x^2+y^2)$. The surface is parameterized over the region $R = \{u^2+v^2 \leq 1\}$ and $\int\int_S \vec{F}\cdot d\vec{S} = \int_0^1 \int_0^{2\pi} 3r^3\ d\theta dr = \boxed{\frac{3\pi}{2}}$.

Problem 14)	(10 points)

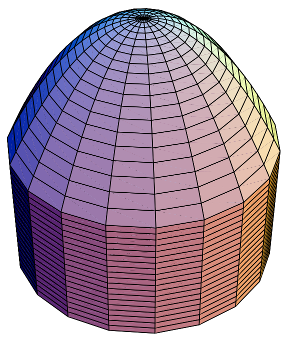
Let S be the graph of the function $f(x,y) = 2-x^2-y^2$ which lies above the disk $\{(x,y) \mid x^2+y^2 \leq 1\}$ in the xy-plane. The surface S is oriented so that the normal vector points upwards. Compute the flux $\int\int_S \vec{F}\cdot d\vec{S}$ of the vectorfield

$$\vec{F} = (-4x + \frac{x^2+y^2-1}{1+3y^2}, 3y, 7-z-\frac{2xz}{1+3y^2})$$

through S using the divergence theorem.

Solution:
We apply the divergence theorem to the region $E = \{0 \leq z \leq f(x,y),\ x^2 + y^2 \leq 1\ \}$. Using $\text{div}(F) = -2$, we get

$$\begin{aligned}\int\int\int \text{div}(F)\ dV &= \int_0^1 \int_0^{2\pi} \int_0^{2-r^2} (-2)\ r\ drd\theta dz \\ &= (-2) \int_0^1 \int_0^{2\pi} (2 - r^2)\ r\ drd\theta dz \\ &= (-2)(2\pi)(2/2 - 1/4) = -3\pi\ .\end{aligned}$$



By the divergence theorem, this is the flux of F through the boundary of E which consists of the surface S, the cylinder S_1 : $r(u,v) = (\cos(u), \sin(u), v)$ with normal vector $r_u \times r_v = (-\sin(u), \cos(u), 0) \times (0,0,1) = (\cos(u), \sin(u), 0)$ plus the flux through the floor $S_2 : \vec{r}(u,v) = (v\sin(u), v\cos(u), 0)$ with normal vector $r_u \times r_v = (0,0,-v)$. The flux through S_1 is

$$\begin{aligned}\int\int_{S_1} \vec{F} \cdot dS &= \int_0^1 \int_0^{2\pi} F(\cos(u), \sin(u), v) \cdot (\cos(u), \sin(u), 0)\ dudv \\ &= \int_0^1 \int_0^{2\pi} (-4\cos^2(u) + 3\sin^2(u))\ dudv = -\pi\end{aligned}$$

The flux through S_2 is

$$\begin{aligned}\int\int_{S_2} \vec{F} \cdot dS &= \int_0^1 \int_0^{2\pi} F(v\sin(u), v\cos(u), 7) \cdot (0,0,-v)\ dudv \\ &= \int_0^1 \int_0^{2\pi} (-7v)\ dudv = -7\pi\end{aligned}$$

By the divergence theorem, $\int\int_S \vec{F} \cdot dS + \int\int_{S_1} \vec{F} \cdot dS + \int\int_{S_2} \vec{F} \cdot dS = -3\pi$ so that $\int\int_S \vec{F} \cdot dS = -3\pi + \pi + 7\pi = \boxed{5\pi}$.