

This is part 2 (of 2) of the weekly homework. It is due August 5'th in class.

SUMMARY.

- $F(x, y) = (P(x, y, z), Q(x, y, z))$ **vector field in the plane.**
- $F(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$ **vector field in space.**
- $F(x, y, z) = \nabla f(x, y, z)$ **gradient field.**
- $F(x, y)$ vector field. **Flow line:** curve $r(t)$ such that $r'(t)$ is parallel to $F(r(t))$. Example: $r(t) = (a \cos(t), a \sin(t))$ are flow lines to $F(x, y) = (-y, x)$.
- $\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t) dt$ **line integral** of F along curve $C : t \mapsto r(t)$.
- Example: $C : r(t) = (\cos(t), \sin(t)), t \in [0, 2\pi]$ (circle), $F(x, y) = (-y, x)$. $\int_C F \cdot dr = \int_0^{2\pi} F(r(t)) \cdot r'(t) dt = \int_0^{2\pi} (-\sin(t), \cos(t)) \cdot (-\sin(t), \cos(t)) dt = \int_0^{2\pi} 1 dt = 2\pi$.
- We say f is **conservative** or f is a **gradient field** if $F(x, y, z) = \nabla f(x, y, z)$.
- $\int_C \nabla f \cdot dr = f(r(b)) - f(r(a))$ **fundamental theorem of line integrals:** by chain rule: $\int_a^b \nabla f(r(t)) \cdot r'(t) dt = \int_a^b \frac{d}{dt} f(r(t)) dt$. Apply the fundamental theorem of calculus.
- If F is conservative in the plane then the line integrals do not depend on the path.
- If F is conservative in the plane and C is a closed curve, then $\int_C F \cdot dr = 0$.

Homework Problems

- 1) (4 points) Determine the gradient field of $f(x, y) = \sin(x + y)$. Draw a contour map of f and sketch also the vector field $F(x, y) = \nabla f$.

Solution:

The level curves of $f(x, y)$ are the same then the level curves of the function $g(x, y) = x + y$ because $\sin(x + y) = c$ means $x + y = \arcsin(c) = C$. The level curves are therefore straight lines.

$\nabla f(x, y) = (\cos(x + y), \cos(x + y))$ is a vector field which is perpendicular to the level curves. It vanishes at places, where $\sin(x + y) = 1$ or $\sin(x + y) = -1$.

- 2) (4 points) Is the vector field $F(x, y) = (P(x, y), Q(x, y)) = (xy, x^2)$ a gradient field?

Solution:

No, because $Q_x = 2x$ and $P_y = x$, this can not be a gradient field $(P, Q) = (f_x, f_y)$.

- 3) (4 points) Let C be the circle $x^2 + y^2 = 16$ and $F(x, y) = (x, y^4)$. Calculate the line integral $\int_C F \cdot dr$.

Solution:

Actually, this is a conservative field because $F(x, y) = \nabla f(x, y)$ with $f(x, y) = x^2/2 + y^5/5$ so that the line integral is zero.

However, we can also compute the line integral: $\vec{r}(t) = (4 \cos(t), 4 \sin(t))$ parametrizes the circle. $\vec{r}'(t) = (-4 \sin(t), 4 \cos(t))$ and $F(\vec{r}(t)) = (4 \cos(t), 256 \sin^4(t))$ so that $\int_0^{2\pi} F(\vec{r}(t)) \cdot \vec{r}'(t) dt = \int_0^{2\pi} (-16 \sin(t) \cos(t) + 1024 \sin^4(t) \cos(t)) dt = (8 \sin^2(t) + 1024 \sin^5(t)/5t)|_0^{2\pi} = 0$.

- 4) (4 points) Let C be the space curve $r(t) = (\cos(t), \sin(t), t)$ for $t \in [0, 1]$ and let $F(x, y, z) = (y, x, 5)$. Calculate the line integral $\int_C F \cdot dr$.

Solution:

$$\int_C f \cdot dr = \int_0^1 (\sin(t), \cos(t), 5) \cdot (-\sin(t), \cos(t), 1) dt = \int_0^1 (\cos(2t) + 5) dt = \sin(2)/2 + 5.$$

- 5) (4 points) Find the work done by the force field $F(x, y) = (x \sin(y), y)$ on a particle that moves along the parabola $y = x^2$ from $(-1, 1)$ to $(2, 4)$.

Solution:

We parametrize the curve C by $r(t) = (t, t^2)$ so that $r'(t) = (1, 2t)$.

$$W = \int_C F \cdot dr = \int_{-1}^2 (t \sin(t^2), t^2) \cdot (1, 2t) dt = \int_{-1}^2 (t \sin(t^2) + 2t^3) dt = \cos(1)/2 - \cos(4)/2 + 15/2.$$

Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

- 1) An electric charge produces the electric field $F = \nabla f$ where $f(x, y, z) = 1/\sqrt{|x^2 + y^2 + z^2|}$. Assume you have three positive electric charges located on an equilateral triangle with side length 1. Without writing down any formula, sketch the field F in the plane spanned by the charges. The field is the sum of the fields of the three charges.
- 2) Consider a O shaped pipe which is filled only on the right side with water. A wooden ball falls on the right hand side in the air and moves up in the water. Why does this "perpetual motion machine" not work?



- 3) What is wrong with the Escher pictures with the stair in which people always walk down or with the waterfall. The figures suggest the existence of a force field which is not conservative. How do the Escher pictures "work"?

