

This is part 1 (of 2) of the weekly homework. It is due August 5 at the beginning of class.

SUMMARY.

- $\iiint_E f \, dV = \int_0^p \int_q^r \int_s^t f(x, y, z) \, dz dy dx$ **triple integral**.
- $V(E) = \iiint_E 1 \, dV$ **volume** of body E .
- $M(E) = \iiint_E \rho \, dV$ **mass** of body E with density ρ .
- $\frac{1}{V(E)} \iiint_E f \, dV$ **average value** of f over E .
- **Spherical coordinates** $\iiint_R g(\rho, \theta, \phi) \rho^2 \sin(\phi) \, d\rho d\theta d\phi$.
- **Cylindrical coordinates** $\iiint_E f(x, y, z) \, dx dy dz = \iiint_R g(r, \theta, z) r \, dr \, d\theta \, dz$.

Homework Problems

- 1) (4 points) Evaluate the iterated integral $\int_0^1 \int_0^z \int_0^{2y} z e^{-y^2} \, dx dy dz$.

Solution:

$$\begin{aligned} \int_0^{2y} z e^{-y^2} \, dx &= 2y z e^{-y^2}. \\ \int_0^z 2y z e^{-y^2} \, dy &= z e^{-z^2} - z. \\ \int_0^1 z e^{-z^2} - z \, dz &= e^{-z^2}/2 - z^2/2 \Big|_0^1 = \boxed{1/(2e)}. \end{aligned}$$

- 2) (4 points) Find the volume of the solid bounded by the paraboloids $z = x^2 + y^2$ and $z = 9 - (x^2 + y^2)$.

Solution:

Use cylindrical coordinates: $\int_0^{\sqrt{9/2}} \int_0^{2\pi} (9 - 2r^2)r \, d\theta dr = -2\pi(9 - 2r^2)^2/8 \Big|_0^{\sqrt{9/2}} = \boxed{81\pi/4}$.

- 3) (4 points) Find the **moment of inertia** $\iiint_E (x^2 + y^2) \, dV$ of a cone $E = \{x^2 + y^2 \leq z^2, 0 \leq z \leq 1\}$, when the cone is rotated around the z -axis.

Solution:

$$2\pi \int_0^1 \int_0^z r^3 \, dr dz = 2\pi \int_0^1 z^4/4 \, dz = \boxed{\pi/10}.$$

- 4) (4 points) Sketch the solid whose volume is given by the integral $\int_0^{\pi/4} \int_0^{\pi/2} \int_0^3 \rho^2 \sin(\phi) \, d\rho d\theta d\phi$. Compute the integral.

Solution:

This is a quarter of a cone.

$$\text{The integral is } (\pi/2) \int_0^3 \rho^2 \, d\rho \int_0^{\pi/2} \sin(\phi) \, d\phi = (\pi/2)9(1 - 1/\sqrt{2}) = \boxed{9\pi(2 - \sqrt{2})/4}.$$

- 5) (4 points) Find the mass of a solid hemi sphere of radius 1 if the density $\rho(x, y, z)$ at any point (x, y, z) is equal to the distance from its base.

Remark. In other words, find $\iiint_E z \, dx dy dz$, where E is the part of the unit sphere with $z \geq 0$.

Solution:

Use spherical coordinates:

$$\int_0^{2\pi} \int_0^{\pi/2} \int_0^1 (\rho \cos(\phi)) \rho^2 \sin(\phi) d\rho d\phi d\theta$$

which evaluates to $2\pi \int_0^1 \rho^3 d\rho \int_0^{\pi/2} \sin(2\phi)/2 d\phi = 2\pi(1/4)(1/2) = \boxed{\pi/4}$.

Remarks

(You don't need to read these remarks to do the problems.)

The **moment of inertia** is defined as $I = \int \int \int_E r^2 \rho dV$ of a body E , where $r = r(x, y, z)$ is the distance of a point (x, y, z) to the axes of rotation and ρ is the density of the body.

The moment of inertia is the rotational analogue to mass .	Sphere	Hollow	$(2/3)MR^2$
	Sphere	Solid	$(2/5)MR^2$
	Cylinder	Solid (i.e. CD!)	$(1/2)MR^2$
	Plate	Axes through center	$(1/12)M(a^2 + b^2)$

If a body rotates with angular velocity ω , then $E = I\omega^2/2$ is the rotational energy of the body. The fact that fast rotation leads to a large energy can be demonstrated by spinning a CD until it disintegrates ...



Don't try that yourself!

Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

- 1) Find the volume of the intersection of two cylinders $y^2 + z^2 = 1$ and $x^2 + z^2 = 1$. Hint. Look what happens, when you cut the body at a fixed z value and calculate the area of this section.
- 2) Use a quadruple integral to find the "hyper volume" of the hyper-sphere $x^2 + y^2 + z^2 + w^2 = r^2$ in \mathbf{R}^4 .

Hint. If you slice up the hyper-sphere at w , you get a sphere of radius $\sqrt{r^2 - w^2}$. Integrate the volume of the sphere from $w = -r$ to $w = r$ using substitution.

- 3) Interpret the triple integral $\int \int \int_E f(x, y, z) dx dy dz$ as the hyper volume of four dimensional region.