This is part 3 (of 3) of the weekly homework. It is due July 29 at the beginning of class.

SUMMARY.

- A vector valued function $\vec{r}(u,v)$ defines a **parametric surface** defined on a region R. It has the **surface area** $\int\int_R |\vec{r}_u(u,v) \times \vec{r}_v(u,v)|\; dudv$.

Homework Problems

1) (4 points) Find the area of the paraboloid $x = y^2 + z^2$ that lies inside the cylinder $y^2 + z^2 = 9$.

Solution:
We use polar coordinates in the yz-plane. The paraboloid is parametrized by $(u,v) \mapsto (v, v^2\cos(u), v^2\sin(u))$ and the surface integral $\int_0^3 \int_0^{2\pi} |\vec{r}_u \times \vec{r}_v|\; dudv$ is equal to $\int_0^3 \int_0^{2\pi} v\sqrt{1+4v^2}\; dudv = 2\pi \int_0^3 v\sqrt{1+4v^2}\; dv = \pi(37^{3/2} - 1)/6$.

2) (4 points) a) You know a formula for the area of a parallelogram with the edge points $(0,0,0)$, $(1,1,0)$,$(0,1,1)$ with $(1,2,1)$ using the cross product. Find the area using this formula.

b) Parametrize the same parallelogram as in a) with a function $\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$, write down the integral for the surface area and evaluate the integral.

Solution:
a) The cross product of $(1,1,0)$ and $(0,1,1)$ is $(1,-1,1)$ which has length $\sqrt{3}$. This is the area of the parallelogram.
b) $\vec{r}(u,v) = u(1,1,0) + v(0,1,1)$. Now $\vec{r}_u(u,v) = (1,1,0)$ and $\vec{r}_v(u,v) = (0,1,1)$ and $|r_u \times r_v| = |(1,-1,1)| = \sqrt{3}$.

3) (4 points) Use the formula for the surface area to compute the area of a cone with a base of radius $r = 3$ and height 5.

Solution:
We parametrize the cone as $\vec{r}(u,v) = (3v\cos(u), 3v\sin(u), 5v)$, where $v \in [0,1]$ and $u \in [0,2\pi]$. We get $\vec{r}_u \times \vec{r}_v = (v\cos(u), v\sin(u), -v)$ which has length $\sqrt{306}v$. The integral is $\sqrt{306} \cdot 2\pi \cdot 1/2 = \pi\sqrt{306} = 3\sqrt{34}\pi$.

4) (4 points) Use the formula for the surface integral to compute the surface area for the points on the sphere of radius 3 for which the Euler angle ϕ is between $\pi/4$ and $\pi/3$ and for which the Euler angle θ is between 0 and $\pi/2$.

Solution:
$\int_{\pi/4}^{\pi/3} \int_0^{\pi/2} 3^2 \sin(\phi)\; d\theta\; d\phi = 9\pi/2(-\cos(\pi/3) + \cos(\pi/4)) = 9\pi/4(\sqrt{2} - 1 : 1)$.

5) (4 points) In a previous homework, you have seen that a torus can be parametrized by $\vec{r}(t) = ((2+\cos(u))\cos(v), (2+\cos(u))\sin(v), \sin(u))$. Find the surface area of the torus.

Solution:
The cross product of $\vec{r}_u = (-(2+\cos(v))\sin(u), (2+\cos(v))\sin(u), \sin(v))$ and $\vec{r}_v = (-\cos(u)\sin(v), -\sin(u)\sin(v), \cos(v))$ is $(\cos(u)\cos(v)(2+\cos(v)), \cos(v)(2+\cos(v))\sin(u), (2+\cos(v))\sin(v))$ which has length $|\vec{r}_u \times \vec{r}_v| = 2+\cos(v)$. The surface integral is $\int_0^{2\pi}\int_0^{2\pi} (2+\cos(v))\; dudv = 8\pi^2$.

Challenge Problems

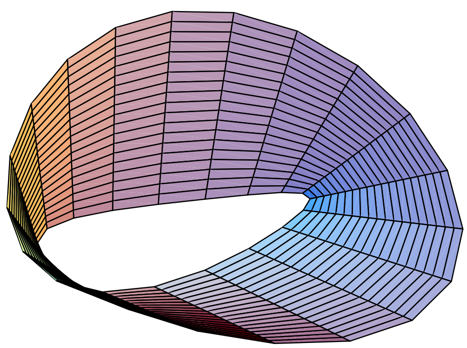
(Solutions to these problems are **not** turned in with the homework.)

1) The Moebius strip is a surface which has only one side. It is parametrized as $(1 + (v - 1/2)\cos(u/2))\cos(u), (1 + (v - 1/2)\sin(u/2))\sin(u), (v - 1/2)\sin(u/2))$. What surface do you compute with the integral

$$\int_0^{2\pi} \int_{-1}^{1} |\vec{r}_u(u,v) \times \vec{r}_v(u,v)| \, dudv \ ?$$

What surface do you compute with the integral

$$\int_0^{4\pi} \int_{-1}^{1} |\vec{r}_u(u,v) \times \vec{r}_v(u,v)| \, dudv \ ?$$



2) In class, you have seen a surface which incloses a finite volume and has infinite surface area. Can you construct for any constant M like $M = 10^{100} cm^2$ a surface inside the unit ball such that the surface area is bigger than M? The picture below should be a hint.