This is part 3 (of 3) of the weekly homework. It is due July 22 at the beginning of class.

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SUMMARY.

- Extremize $f(x,y)$ under the constraint $g(x,y) = c$: Solve $g(x,y) = c, \nabla f(x,y) = \lambda \nabla g(x,y)$ with **Lagrange multiplier** $\lambda$, These are 3 equations for 3 unknowns $x, y, \lambda$:

$$
\begin{aligned}
f_x(x,y) &= \lambda g_x(x,y) \\
f_y(x,y) &= \lambda g_y(x,y) \\
g(x,y) &= c
\end{aligned}
$$

In three dimensions, the Lagrange equations form 4 equations for 4 unknowns.

$$
\begin{aligned}
f_x(x,y,z) &= \lambda g_x(x,y,z) \\
f_y(x,y,z) &= \lambda g_y(x,y,z) \\
f_z(x,y,z) &= \lambda g_z(x,y,z) \\
g(x,y,z) &= c
\end{aligned}
$$

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# Homework Problems

1) (4 points) Find the extrema of the function $f(x,y) = e^{-x^2-y^2}(x^2 + 2y^2)$ on the circle $g(x,y) = x^2 + y^2 = 4$ using the method of Lagrange multipliers.

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**Solution:**
The Lagrange equations are

$$
\begin{aligned}
(2x - 2x^3 - 4xy^2)e^{-x^2-y^2} &= \lambda 2x \\
(4y - 4y^3 - 2x^2y)e^{-x^2-y^2} &= \lambda 2y \\
x^2 + y^2 &= 4 .
\end{aligned}
$$

Case 1: If $x = 0$, then the first equation is ok and we get from the third equation $y = \pm 2$.
Case 2: If $y = 0$, then the second equation is ok and we get from the third edquation $x = \pm 2$.
Case 3: If $x = 0$ and $y = 0$, then the first two equations are ok, but clashes with the third. Forget this case.
Case 4: If both $x$ and $y$ are not zero we can divide the first equation by $2x$ and the second by $2y$. We also replace $-x^2 - y^2$ by $-4$

$$
\begin{aligned}
(1 - x^2 - 2y^2)e^{-4} &= \lambda \\
(2 - 2y^2 - x^2)e^{-4} &= \lambda \\
x^2 + y^2 &= 4 .
\end{aligned}
$$

But setting the first two equations equal leads to a contradiction. Also this case 4) has no solutions. We end up with the four solutions $(2,0), (-2,0), (0,2), (0,-2)$. The minimal values are $f(\pm 2, 0) = 4e^{-4}$, the maximal values are $f(0, \pm 2) = 8e^{-4}$.

2) (4 points) Find the extrema of the same function $f(x, y) = e^{-x^2-y^2}(x^2 + 2y^2)$ as in the previous problem but now on the entire disc $\{x^2 + y^2 \le 4\}$ of radius 2.

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**Solution:**

In the last homework we have seen that the equation $\nabla f = ((2x - 2x^3)e^{-x^2-y^2}, (4y - 4y^3)e^{-x^2-y^2}) = (0,0)$ has the solutions $x = 0, y = 0$, $x = 0, y = \pm 1$, $y = 0, x = \pm 1$. Together with the previous problem, we can now make a list of all the candidates for extrema

1. Extrema inside

| point | $f =$ |
|-------|-------|
| (0,0) | 0 |
| (1,0) | 1/e |
| (-1,0) | 1/e |
| (0,1) | 2/e |
| (0,-1) | 2/e |

2. Extrema on the boundary

| point | $f =$ |
|-------|-------|
| (2,0) | $2/e^4$ |
| (-2,0) | $2/e^4$ |
| (0,2) | $4/e^4$ |
| (0,-2) | $4/e^4$ |

We see that the origin is the minimum and the points $(0, \pm 1)$ are both the maxima.

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3) (4 points) Find the points $(x, y, z)$ on the surface $g(x, y, z) = xy^2 - z^3 - 2 = 0$ that are closest to the origin $(0, 0, 0)$.

**Solution:**
Instead of extremizing the distance $\sqrt{x^2 + y^2 + z^2}$ we extremize the function $f(x, y) = x^2 + y^2 + z^2$. We have the Lagrange equations

$$\begin{aligned} 2x &= \lambda y^2 \\ 2y &= \lambda 2xy \\ 2z &= -\lambda 3z^2 \\ xy^2 &= z^3 + 2 \end{aligned}$$

1. Case: $z = 0$. We can then not have $y = 0$ nor $x = 0$ and end up with

$$\begin{aligned} 2x &= \lambda y^2 \\ 2 &= \lambda 2x \\ xy^2 &= 2 \end{aligned}$$

which gives $2 = \lambda^2 y^2$, $x^2 = \lambda$ and so that $x = 1, y = \pm\sqrt{2}$. 2. Case: $x = 0$. Gives $y = 0$ and $z = -2^{1/3}$.
3. Case: $y = 0$. Gives $x = 0$ and $z = -2^{1/3}$.
4. Case: all $x, y, z$ are nonzero. Then

$$\begin{aligned} 2x &= \lambda y^2 \\ 1 &= \lambda x \\ 2 &= -\lambda 3z \\ xy^2 &= z^3 + 2 \end{aligned}$$

Eliminating $\lambda = 1/x$ gives

$$\begin{aligned} 2x^2 &= y^2 \\ 2 &= -3z/x \\ xy^2 &= z^3 + 2 \end{aligned}$$

Solving for $y^2 = 2x^2$ and $z = -2x/3$ from the first two equations and plugging this into the third gives $2x^3 = -8x^3/27 + 2$ gives $x = 3/31^{(1/3)}$ and so $y = \pm\sqrt{2}(3/31^{(1/3)})$. The distance from the point $(0, 0, -2^{(1/3)})$ to the origin is $2^{(1/3)} \sim 1.25$ the distance from the points $(3/31^{1/3}(1, \pm\sqrt{2}, -2/3)$ to the origin is $31^{1/6} \sim 1.7723$. The first one is the minimum.

4) (4 points) Let $a, b, c$ be non-negative constants and let $F$ be the function $F(x, y, z) = -x\log(x) - y\log(y) - z\log(z) - ax - by - cz$. Find the maxima and minima of $F$ on $x > 0, y > 0, z > 0$ under the constraint $x + y + z = 1$.

**Solution:**
The Lagrange equations are

$$
\begin{aligned}
-\log(x) - 1 - a &= \lambda \\
-\log(x) - 1 - b &= \lambda \\
-\log(x) - 1 - c &= \lambda \\
x + y + z &= 1
\end{aligned}
$$

From the first three equations, we get

$$
\begin{aligned}
x = e^{-(1+a+\lambda)} &= e^{-1-\lambda}e^{-a} \\
y = e^{-(1+b+\lambda)} &= e^{-1-\lambda}e^{-b} \\
z = e^{-(1+c+\lambda)} &= e^{-1-\lambda}e^{-c}
\end{aligned}
$$

Plugging this into the fourth equation gives $e^{-1-\lambda}(e^{-a} + e^{-b} + e^{-c}) = 1$ so that $e^{-1-\lambda} = (e^{-a} + e^{-b} + e^{-c})^{-1}$ and

$$
\begin{aligned}
x &= e^{-a}/(e^{-a} + e^{-b} + e^{-c}) \\
y &= e^{-b}/(e^{-a} + e^{-b} + e^{-c}) \\
z &= e^{-c}/(e^{-a} + e^{-b} + e^{-c})
\end{aligned}
$$

5) (4 points) Which pyramid of height $h$ over a square $[-a, a] \times [-a, a] = \{(x, y) \mid -a \leq x \leq a, -a \leq y \leq a$ and surface area 4 has maximal volume?

**Solution:**
The area is $4a\sqrt{h^2 + a^2} + 4a^2 = 4$, the volume is $V = ha^2/3$. We have the mathematical problem to extremize $f(x, y) = yx^2$ over the constraint $g(x, y) = x\sqrt{y^2 + x^2} + x^2 = 1$. The Lagrange system is

$$
\begin{aligned}
2xy &= \lambda(\sqrt{y^2 + x^2} + x^2/\sqrt{y^2 + x^2} + 2x) \\
x^2 &= \lambda yx/\sqrt{y^2 + x^2} \\
1 &= x\sqrt{y^2 + x^2} + x^2
\end{aligned}
$$

There are different possibilities to solve this a bit tricky system. Since $x \neq 0$, we can divide the second equation by $x$. replace it by $x\sqrt{x^2 + y^2} = \lambda y$ or its square. The third equation becomes therefore $1 = \lambda y + x^2$. The square roots of the first equation can then also be replaced by $\lambda y/x$.

$$
\begin{aligned}
2xy &= \lambda(\lambda y/x + x^3/(y\lambda) + 2x) \\
x^2(y^2 + x^2) &= \lambda^2 y^2 \\
1 &= \lambda y + x^2 \ .
\end{aligned}
$$

Until now, we were just trying to simplify, not yet eliminiating. To eliminate $\lambda$, we multiply the first equation by $\lambda$ and replace $\lambda y$ by $(1-x^2)$. We end up with two equations:

$$
\begin{aligned}
2xy^2 &= (1 - x^2)((1 - x^2)/x + x^3/(1 - x^2) + 2x) \\
x^2(y^2 + x^2) &= (1 - x^2)^2
\end{aligned}
$$

Te second equation is equivalent to $x^2 y^2 = 1 - 2x^2$ so that the second equation gives $y^2 = (1 - 2x^2)/x^2$. This can be plugged into the first equation to obtain

$$
2x(1 - 2x^2) = x(1 - x^2)^2 + x^5 + 2x^3(1 - x^2)
$$

which simplifies to $x = 4x^3$. The solution is therefore $a = 1/2$, $h = \sqrt{2}$. Whao!

**Remark**. This problem was on the edge what is possible to solve by hand. A two dimensional version of the problem, where one asks for the equilateral triangle of height $h$ and ground side $2a$ which has maximal area if the circumfence $2a + 2\sqrt{a^2 + h^2} = 2$ is fixed. The solutions of the Lagrange equations are then $a = 1/3, h = 1/\sqrt{3}$.

# Remarks

(You don't need to read these remarks to do the problems.)

Remark to problem 4) This problem appears in thermodynamics and is relevant in biology or chemistry. If $x, y, z$ are the probabilities that a system is in state $X, Y, Z$ and $a, b, c$ are the energies for these states. Then $-x \log(x) - y \log(y) - z \log(z)$ is called the **entropy** of the system and $E = ax + by + cz$ is the **energy**. The number $F(x, y, z)$ is called the **free energy**. If energy is fixed, nature tries to maximize entropy. Otherwise it tries to **minimize the free energy** $F = S - E$. If we extremize $F$ under the constraint of having total probability $G(x, y, z) = x + y + z = 1$, we obtain the so called **Gibbs distribution**.

# Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

1) What does it mean that the Lagrange multiplier $\lambda$ is zero in a constrained optimization problem?

2) Extend the Lagrange method to arbitrary dimensions. Find the equations to find the extrema of a function $f(x_1, ..., x_n)$ under the constraint $g(x_1, ..., x_n) = c$.

3) Let $I = -\sum_{i=1}^{n} p_i \log(p_i)$ be the entropy of a probability distribution $(p_1, \ldots, p_n)$. Show that among all probability distributions, the one where $p_i = 1/n$ is the one which maximizes entropy.

4) Find the box of dimensions $x, y, z, w$ in four dimensional space which has maximal "hyper volume" $V = xyzw$ under the constraint that the "surface volume" $A = 2xy + 2xz + 2xw + 2yz + 2yw + 2zw$ is fixed equal to 12.