

This is part 2 (of 3) of the homework which is due July 8 at the beginning of class.

SUMMARY.

- $f(x, y), g(x, y, z)$ **functions of several variables**.
- $\{f(x, y) = c\}$ describes **curve** called **contour curve**.
- $\{g(x, y, z) = d\}$ describes **surface** called **contour surface**.
- $\{(x, y, z) = f(x, y)\}$ **graph** of function $f(x, y)$. It is a surface.
- **trace**: intersection of graph with coordinate planes.
- **intercepts**: intersection of graph with coordinate axes.

Quadrics:

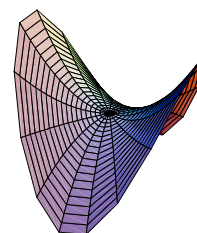
- $x^2 + y^2 + z^2 = 1$ **sphere**.
- $x^2 + y^2 - z^2 = 1$ **one-sheeted hyperboloid**.
- $x^2 + y^2 - z^2 = -1$ **two-sheeted hyperboloid**.
- $x^2 + y^2 - z^2 = 0$ **cone**.
- $x^2 + y^2 = z$ **paraboloid**.
- $x^2 + y^2 = 1$ **cylinder**.

Homework Problems

- 1) (4 points) Let $f(x, y) = x^2 - y^2$.
- (1) Write the graph of f as a contour surface $g(x, y, z) = 0$.
 - (1) Find the equations for the three traces of f .
 - (2) Sketch the graph of f .

Solution:

- $z = f(x, y)$ gives $g(x, y, z) = z - f(x, y) = z - x^2 + y^2 = 0$.
- The xy -trace: $x^2 - y^2 = 0$ is a union of two lines.
The yz -trace: $z + y^2 = 0$ is a parabola.
The xz -trace: $z - x^2 = 0$ is a parabola.



- The surface is called a hyperbolic paraboloid, a "pringle".

- 2) (4 points) Consider the surface $z^2 + x^2 - 2x - y = 0$. Draw the three traces. What surface is it?

Solution:

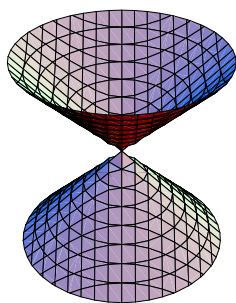
Completion of the square gives $z^2 + (x-1)^2 - y = 1$. The surface is a paraboloid rotational symmetric parallel to the y axes. To see this, it is helpful to draw the generalized traces obtained by intersecting with $y = c$ which gives circles. The other two traces are parabola.

- 3) (4 points) Surfaces satisfying the implicit equation $x^k + y^k = z^k$ with integer k are called **Fermat** surfaces. In this exercise, we don't want you to compute, but to draw and visualize. Traces help you to get the picture right.

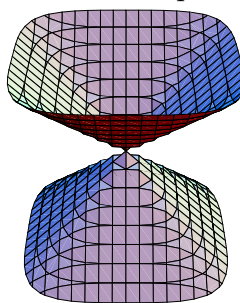
- a) Sketch the Fermat surface for $k = 2$ with traces.
b) Sketch the Fermat surface for $k = 4$ with traces.

Solution:

The surface $x^2 + y^2 = z^2$ is a cone.



The surface $x^4 + y^4 = z^4$ has on each height z a trace which has the shape when you deform a circle to a square.



- 4) (4 points)

- a) (1) Sketch the graph of the function $f(x, y) = 1/(x^2 + 4y^2)$.
b) (1) Sketch the graph of the function $g(x, y) = |x| + |y|$.
c) (1) Sketch some contour curves $f(x, y) = c$ of f .
d) (1) Sketch some contour curves $g(x, y) = c$ of g .

Solution:

- a) A singular spike of elliptic shape.
b) In each quadrant, the graph is linear and a plane. These four pieces come together above the coordinate axes.
c) These are ellipses.
d) These are romboids, by 45 degree turned squares.

- 5) (4 points) Verify that the line $\vec{r}(t) = (1, 3, 2) + t(1, 2, 1)$ is contained in the surface $y = z^2 - x^2$.

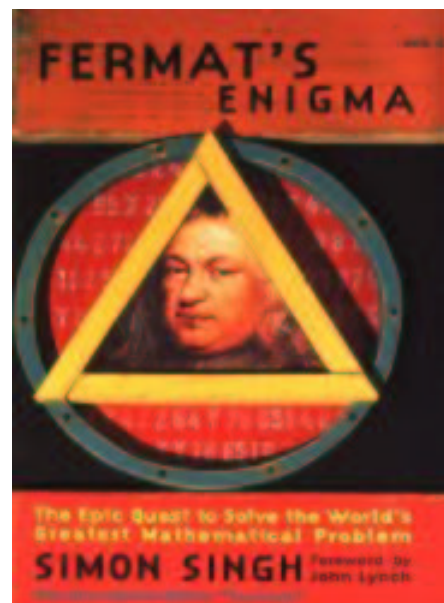
Solution:

Plut in $x = 1 + t, y = 3 + 2t, z = 2 + t$ into the equation $(3 + 2t) = (2 + t)^2 - (1 + t)^2$. The surface is actually a hyperbolic paraboloid.

Remarks

(You don't need to read these remarks to do the problems.)

To problem 3: You have found integer points (x, y, z) lying on the Fermat surface $x^2 + y^2 = z^2$ in a previous homework. It was Fermat, who conjectured first, that there are no nontrivial lattice points on the Fermat surfaces for $n > 2$. This claim is now a theorem. It has been proven only a few years ago. The story about Fermat Last Theorem is told in many places. A nice little book is "The Fermat Enigma" by Simon Singh.



Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

- 1) Find as many curves as you can which are obtained by intersecting two quadrics. To get full credit for this problem you have to provide at least three different curves.
- 2) How would you visualize the graph of a function $f(x, y, z)$ of three variables? How would you describe the set $\{(x, y, z, u) \mid g(x, y, z, u) = d\}$, where g is a function of 4 variables and d is a constant? Take the example of the four-dimensional sphere $x^2 + y^2 + z^2 + u^2 = 1$.