This is part 3 (of 3) of the homework which is due July 1 at the beginning of class.

SUMMARY.

- $\vec{v} \times \vec{w} = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1)$ **cross product**.
- $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}|\sin(\phi)$, where ϕ is the **angle** between vectors.

  This is the area of parallelogram spanned by $\vec{v}$ and $\vec{w}$.
- $\vec{v} \times \vec{w}$ is **orthogonal** to $\vec{v}$ and to $\vec{w}$ with length $|\vec{v}||\vec{w}|\sin(\phi)$
- $\vec{u} \cdot (\vec{v} \times \vec{w})$ **triple scalar product**, volume of parallelepiped spanned by $\vec{u}, \vec{v}, \vec{w}$.

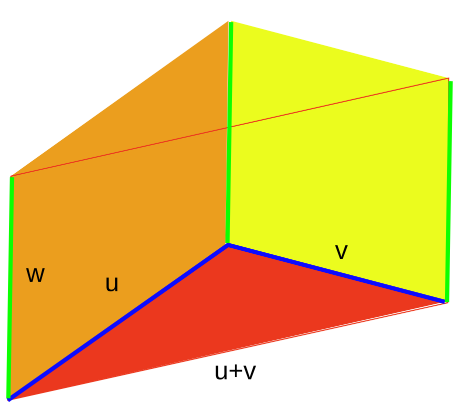
Homework Problems

1) (4 points)
a) (2) Find a the cross product $\vec{w}$ of $\vec{u} = (-2, -1, 2)$ and $\vec{v} = (-2, -2, 3)$.
b) (1) Find a unit vector $\vec{n}$ orthogonal to $\vec{u}$ and $\vec{v}$.
c) (1) Find the volume of the parallelepiped spanned by $\vec{u}, \vec{v}$ and $\vec{w}$.

Solution:
a) $\vec{u} \times \vec{v} = \vec{w} = (1, 2, 2)$
b) $|\vec{w}| = 3$, $\vec{n} = \vec{w}/|\vec{w}| = (1/3, 2/3, 2/3)$.
c) $\vec{w} \cdot (\vec{u} \times \vec{v}) = \vec{w} \cdot \vec{w} = |\vec{w}|^2 = 9$.

2) (4 points) Draw a picture which explains (if possible without words) why the distributivity law $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$ is true for vectors $\vec{u}, \vec{v}, \vec{w}$ which lie in one plane Remark: to clarify your picture, you can add some explanation of course.

Solution:



3) (4 points) a) Verify the identity $|\vec{v} \times \vec{w}|^2 = |\vec{v}|^2|\vec{w}|^2 - (\vec{v} \cdot \vec{w})^2$ from the lecture.
b) Knowing $(\vec{v} \cdot \vec{w}) = |\vec{v}||\vec{w}|\cos(\phi)$ derive $|(\vec{v} \times \vec{w})| = |\vec{v}||\vec{w}|\sin(\phi)$

Solution:
a) Direct, but a bit tedious computation. b) Use the identity $\cos^2(x) + \sin^2(x) = 1$.

4) (4 points)
a) ("Enter the matrix") Given two vectors $\vec{v}, \vec{w}$, and the vectors $\vec{i} = (1,0,0), \vec{j} = (0,1,0), \vec{k} = (0,0,1)$, we write down the following 3×3 array called a "matrix":

$$\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} .$$

Can you find a handy algorithm to find the cross product of $\vec{v}$ and $\vec{w}$ using this matrix?
b) ("The Matrix reloaded") Given three vectors $\vec{u}, \vec{v}, \vec{w}$, we write down the following "matrix":

$$\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} .$$

Describe an easy to remember algorithm to find the triple product $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$ using this matrix?

Solution:
a) The cross product is $\vec{i}(v_2w_3 - v_3w_2) + \vec{j}(v_3w_1 - v_1w_3) + \vec{k}(v_1w_2 - v_2w_1)$. If we take the product along the three diagonals and subtract from it the sum of the products of the three anti-diagonals, we obtain the cross product. This sum is called the **determinant** of the matrix.
b) Again, this is the determinant of the matrix. First we sum the products over the three diagonals, then we subtract the sum of the products along the three anti-diagonals:

$$\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} - \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

5) (4 points) Given three vectors $\vec{u}, \vec{v}, \vec{w}$ with $V = \vec{u} \cdot (\vec{v} \times \vec{w}) \neq 0$. Define three new vectors

$$\begin{aligned} \vec{a} &= (\vec{v} \times \vec{w})/V \\ \vec{b} &= (\vec{w} \times \vec{u})/V \\ \vec{c} &= (\vec{u} \times \vec{v})/V . \end{aligned}$$

Verify that $\vec{a} \cdot (\vec{b} \times \vec{c}) = 1/V$.

Hint. You can use the identity $\vec{b} \times (\vec{u} \times \vec{v}) = (\vec{b} \cdot \vec{v})\vec{u} - (\vec{b} \cdot \vec{u})\vec{v}$ which holds in general. (If you have time, derive this identity, but this is not required to get full credit for this problem).

Solution:
Focus first on $(\vec{b} \times \vec{c}) = \vec{b} \times (\vec{u} \times \vec{v})/V$ and use the hint to get $(\vec{b} \cdot \vec{v})\vec{u}/V - (\vec{b} \cdot \vec{u})\vec{v}/V$. The problem asks for the dot product of this with $\vec{a}$. Now, since $\vec{a}$ is orthogonal to the second term, we obtain $\vec{a}(\vec{b} \cdot \vec{v}) \cdot \vec{u}/V$. When plugging in the definitions of $\vec{a}$ and $\vec{b}$, we are left with $V^2/V^3 = 1/V$.

Remarks

(You don't need to read these remarks to do the problems.)

To problem 5): three vectors whose triple scalar product does not vanish are called **non-coplanar** . Adding integer multiples of such vectors form a **lattice**. The points of the lattice are all points $n\vec{u} + m\vec{v} + k\vec{w}$, where n, m, k are integers.
The three new vectors $\vec{a}, \vec{b}$ and $\vec{c}$ defined in problem 5) define a new lattice which is called the **reciprocal lattice**. Crystallographers also denote them by $\vec{u}^*, \vec{v}^*$ and $\vec{w}^*$. What you have shown in 5) is that the volume V^* of the unit cells of the reciprocal lattice is the inverse $1/V$ of the volume V of the unit cell of the lattice itself. The reciprocal lattice is essential for the study of crystal lattices and their diffraction properties which can be measured by shooting X-rays onto them. A convenient way to link the structure of the material to it's diffraction pattern is through the reciprocal lattice.

Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

1) Find a general formula for the volume of a tetrahedron with corners P, Q, R, S.
Hint. Find first a formula for the area of one of its triangular faces, and then a formula for the distance from the fourth point to that face.

2) The change of the angular momentum $\vec{L}$ in the torque satisfies $\frac{d}{dt}\vec{L} = \vec{L} \times \vec{\Omega}$, where $\vec{\Omega}$ is the angular velocity vector. Verify that the length of $\vec{L}$ does not change in time.