Solutions to Problem Set 3

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Part 1

1.$\nabla f(x,y) = (2x, 18y)$. The level sets are ellipses and the gradient is perpendicular to these ellipses pointing away from the origin.

2.

$$\begin{aligned}
\frac{\partial P}{\partial V} &= \frac{\partial}{\partial V}(\frac{mRT}{V}) = -\frac{mRT}{V^2} \\
\frac{\partial V}{\partial T} &= \frac{\partial}{\partial T}(\frac{mRT}{P} = \frac{mR}{P} = \frac{mR}{\frac{mRT}{V}} = \frac{V}{T} \\
\frac{\partial T}{\partial P} &= \frac{\partial}{\partial P}(\frac{PV}{mR}) = \frac{V}{mR}
\end{aligned}$$

So $\frac{\partial P}{\partial V} * \frac{\partial V}{\partial T} * \frac{\partial T}{\partial P} = -\frac{mRT}{V^2} * \frac{V}{T} * \frac{V}{mR} = -1$.

3. $f(x,t) = e^{-rt}f(x+ct)$. So, $f_x = e^{-rt}f'(x+ct)$ and $f_t = -re^{-rt}f(x+ct) + ce^{-rt}f'(x+ct) = -rf + cf_x$.

4. $P(L,K) = bL^{.75}K^{.25}$.

$$\begin{aligned}
P_L &= .75bL^{-.25}K^{.25} \\
P_K &= .25bL^{.75}K^{-.75} \\
LP_L + KP_K &= .75bL^{.75}K^{.25} + .25bL^{.75}K^{.25} = P
\end{aligned}$$

5.

$$\begin{aligned}\nabla f &= \frac{.5}{\sqrt{10-x^2-5y^2}}(-2x,-10y)\\ &= (-x,-5y)\end{aligned}$$

Thus, $\nabla f(2,1) = (-2,-5)$. Since $f(2,1) = \sqrt{10-4-5} = 1$ the linearization of f about $(2,1)$ is $L(x,y) = 1-2(x-2)-5(y-1)$. Using the linearization we can estimate $f(1.95,1.04) \approx L(1.95,1.04) = 1+2*.05+5*.04 = 1+.1-.2 = .9$.

Part 2

1. (First Approach) $T(r(t)) = (\cos(t))^2 + (\sin(t))^2 + 2t^2 = 1+2t^2$. $\frac{d}{dt}T(r(t)) = 4t$ so $\frac{d}{dt}T(r)(0) = 0$.

(Second Approach) $\nabla T = (2x,2y,4z)$, $r(0) = (1,0,0)$, and $r'(0) = (0,1,1)$ so

$$\frac{d}{dt}T(r)(0) = \nabla T(1,0,0)\cdot r'(0) = (2,0,0)\cdot(0,1,1) = 0$$

2. The surface is a level set for the function $f(x,y,z) = x^2+2y^2-z^2$ so the tangent plane is perpendicular to the gradient of f. $\nabla f = (2x,4y,-2z)$ so $\nabla f(2,1,0) = (4,4,0)$. Thus, the tangent plane is $4(x-2)+4(y-1) = 0$.

3. $\nabla f(x,y) = (2x+y\cos(xy), x\cos(xy))$ so $\nabla f(1,0) = (2,1)$. If $u = (u_1,u_2)$ is a unit vector such that the derivative of f in the direction of u is 1 then $u_1^2+u_2^2 = 1$ and $2u_1+u_2 = 1$. Substituting the second constraint on u into the first we see

$$\begin{aligned}u_2 &= 1-2u_1\\ u_1^2+(1-2u_1)^2 &= 1\\ u_1^2+1-4u_1+4u_1^2 &= 5u_1^2-4u_1+1 = 1\\ \text{so } u_1(5u_1-4) = 0 &\end{aligned}$$

Thus $u = (0,1)$ or $(\frac{4}{5},-\frac{3}{5})$.

Part 3

2.$f(x,y) = e^(- x^2 - y^2)(x^2 + 2y^2)$.

$$\begin{aligned} \nabla f(x,y) &= (-2x * e^(- x^2 - y^2)(x^2 + 2y^2) + 2x * e^(- x^2 - y^2), -2y * e^(- x^2 - y^2)(x^2 + 2y^2) - 4y * \\ &= e^(- x^2 - y^2)(x(-2x^2 + 4y^2 + 2), y(-2x^2 + 4y^2 - 4)) \end{aligned}$$

So the critical points solve the simultaneous equations $x(-2x^2 + 4y^2 + 2) = 0$ and $y(-2x^2 + 4y^2 - 4) = 0$. These equations can be solved by breaking the problem down into subcases.

Case 1 $x \neq 0$ and $y \neq 0$ This case allows you to divide the first equation by x and the second equation by y. So you get

$$\begin{aligned} -2x^2 + 4y^2 + 2 &= 0 \\ -2x^2 + 4y^2 - 4 &= 0. \end{aligned}$$

But subtracting the first equation from the second equation gives $-6 = 0$, which, last time I checked, is false. So we have a contradiction and can rule this case out.

Case 2 $x = 0$

This makes the first equation useless but allows you to reduce the second equation to $y(y^2 - 1) = 0$ so $(0,0)$, $(0,1)$, and $(0,-1)$ are all canidates for extrema.

Case 3 $y = 0$ This is just like case 2 and yields the points $(0,0)$, $(1,0)$ and $(-1,0)$

By inspection we can see that $(0,0)$ is a local minimum and that $(1,0)$, $(-1,0)$, $(0,-1)$, $(0,1)$ are local maximum.