Solutions to Problem Set 3

July 23, 2002

Part 1

1.a.

$$\begin{aligned} r(t) &= (t^3, t^2) \\ r'(t) &= (3t^2, 2t) \\ r''(t) &= (6t, 2) \end{aligned} \tag{1}$$

The curve looks like a parabola but with the opposite concavity so it has a cusp at the origin.

b.

$$\begin{aligned} r(t) &= (\sin(t), t^3, \cos(t)) \\ r'(t) &= (\cos(t), 3t^2, -\sin(t)) \\ T(t) &= \frac{1}{\sqrt{1+9t^4}}(\cos(t), 3t^2, -\sin(t)) \end{aligned} \tag{2}$$

The curve looks like a spiral in the direction of the y axis that has been compressed near the origin and streched out at infinity.

2.

$$\begin{aligned} r''(t) &= (\cos(t), -\cos(3t), .2t) \\ r'(t) &= (\sin(t), -\frac{1}{3}\sin(3t), .1t^2) \\ r(t) - r(0) &= (-\cos(t), \frac{1}{9}\cos(3t), \frac{1}{30}t^3) \end{aligned} \tag{3}$$

3. $r(t) = (t^2, 1+t, 1+t^3)$. $r(-1) = (1,0,0)$, so $(1,0,0)$ lies on the curve. As for the point $(1,2,-2)$, $x = 1 \Rightarrow t^2 = 1 \text{thus} t = 1 \text{or} -1$. $y = 2 \Rightarrow 1+t = 2 \text{thus} t = 1$. But, $z = -2 \Rightarrow 1+t^3 = -2 \text{thus} t = -(3)^{\frac{1}{3}} \text{or} 3^{\frac{1}{3}}$. This yields a contradiction, so $(1,2,-2)$ does not lie on the curve.

$$\begin{aligned} r'(t) &= (2t, 1, 3t^2) \\ r''(t) &= (2, 0, 6t) \\ \text{so } r'(-1) = (-2,1,3) & \quad r''(-1) = (2,0,-6) \end{aligned} \tag{4}$$

4.

$$\begin{aligned} x^2 + y^2 = t^2(\cos(t))^2 + t^2(\sin(t))^2 &= t^2((\cos(t))^2 + (\sin(t))^2) \\ &= t^2 * 1 \\ &= t^2 = z \end{aligned} \tag{5}$$

Thus, the curve is a spiral painted on the paraboloid $x^2 + y^2 = z$.

5. $x(t) = 2\cos(t)$ and $y(t) = 2\sin(t)$ so $z(t) = 4\cos(t)\sin(t) = 2\sin(2t)$. $r'(t) = (-2\sin(t), 2\cos(t), 4\cos(2t))$.

Part 2

1. $r'(t) = (2t, t\sin(t), t\cos(t))$. So, $|r'(t)| = \sqrt{4t^2 + t^2} = \sqrt{5}t$ for $t > 0$.

$$\int_0^\pi \sqrt{5}t \mathrm{d}t = \frac{\sqrt{5}}{2}\pi^2$$

2.$r'(t) = (2t, 2, \frac{1}{t})$. So, $|r'(t)| = \sqrt{4t^2 + 4 + (\frac{1}{t})^2} = \sqrt{(2t + \frac{1}{t})^2} = 2t + \frac{1}{t}$ for $t > 0$.

$$\int_1^e 2t + \frac{1}{t}\mathrm{d}t = t^2 + \ln(t)|_1^e = e^2 - 1 + 1 - 0 = e^2$$

Note that in the original problem the lower limit was 0. For this case the arc length is ∞.

3.

$$\begin{aligned} r'(t) &= (e^t\cos(t) - e^t\sin(t), e^t\cos(t) + e^t\sin(t), 1) \\ r''(t) &= (2e^t\sin(t), 2e^t\cos(t), 0) \\ \text{so} r'(0) = (1,1,1) \quad \text{and} \quad & r''(0) = (0,2,0) \end{aligned} \tag{6}$$

$$|r'(0)\text{x}r''(0)| = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 0 & 2 & 0 \end{vmatrix} = |(-2,0,2)| = 2\sqrt{2}$$

Thus, $K(0) = \frac{\sqrt{24}}{9}$.

4.

$$\begin{aligned} x &= u \\ y &= v \\ z &= -\sqrt{-2u^2 - 4v^2 + 1} \end{aligned} \tag{7}$$

One could also use the parametrization,

$$\begin{aligned} x &= \sqrt{\frac{1}{2}}\cos(u)\sin(v) \\ y &= \frac{1}{2}\sin(u)\sin(v) \\ z &= \cos(v) \end{aligned} \tag{8}$$

where $0 \leq u < 2\pi$ and $\frac{\pi}{2} \leq v \leq \pi$.

5.

$$(x,y,z) = ((2+\cos(u))\cos(v), (2+\cos(u))\sin(v), \sin(u))$$

where $0 \leq u < 2\pi$ and $0 \leq v < 2\pi$.