Solutions to Problem Set 1

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Part 1

1. $x + 2y = 4$ is a plane.

2. a. The distance between $P = (3, 7, 5)$ and the z-axis is the distance between P and the closest point on the z-axis, which is $(0, 0, 5)$. So,

$$\text{dist}(P, \text{z-axis}) = \sqrt{3^2 + 7^2} = \sqrt{58}$$

b. This is just the absolute value of the z-coordinate. ans $= 5$.

c. $\sqrt{3^2 + 7^2 + 5^2}$

3. Consider the equation for the sphere with radius $\sqrt{7}$ and center $(6, 5, -2)$.

$$(x - 6)^2 + (y - 5)^2 + (z + 2)^2 = 7.$$

The trace in xy plane is achieved by setting $z = 0$ in this equation. Similarly, the xz and yz traces are achieved by setting $y = 0$ and $x = 0$, respectively.

So for the xy trace consider,

$$
\begin{aligned}
(x - 6)^2 + (y - 5)^2 + 2^2 &= 7 \\
(x - 6)^2 + (y - 5)^2 &= 3.
\end{aligned}
\tag{1}
$$

Thus, the xy trace is a circle with center $(6, 5, 0)$ and radius $\sqrt{3}$.

For the xz trace consider,

$$
\begin{aligned}
(x - 6)^2 + 5^2 + (z + 2)^2 &= 7 \\
(x - 6)^2 + (z + 2)^2 &= -18.
\end{aligned}
$$

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The left side of the equation is positive, where as, the right side is negative. Thus, there is a contradiction and the xz trace is empty. A similar calculation shows that the yz trace is empty.

4.

$$4x^2 + 4y^2 + 4z^2 - 8x + 16y = 1$$
$$4(x^2 - 2x) + 4(y^2 + 4y) + 4z^2 = 1$$
$$4(x^2 - 2x + 1) + 4(y^2 + 4y + 4) + 4z^2 = 1 + 4 + 16$$
$$4(x - 1)^2 + 4(y + 2) + 4z^2 = 21$$
$$(x - 1)^2 + (y + 2)^2 + z^2 = \frac{21}{4}$$

So this is the equation for a sphere of radius $\frac{\sqrt{21}}{2}$ and center $(1, -2, 0)$.

5. $4 < x^2 + y^2 + z^2 < 9$. The inequalities are strict because we want the points that are between but not on the spheres.

Part 2

1. $a + b = (-1, 2) + (5, 3) = (4, 5)$. $a - b = (-6, -1)$.

2. $|v| = \sqrt{2^2 + 4^2 + 2^2} = \sqrt{24}$. So the vector with length 6 and the same direction as v will be $6\frac{v}{|v|} = (-\sqrt{6}, 2\sqrt{6}, \sqrt{6})$.

3. Two vectors are orthogonal when their dot product is zero. Thus, $v \cdot w = (-6, b, 2) \cdot (b, b^2, b) = -6b + b^3 + 2b = b^3 - 4b = 0$. Factoring the equation we see, $b(b - 2)(b + 2) = 0$. $b = 0$ gives the zero vector. Whether the zero vector is orthogonal to v or whether the word orthogonal connotes non-zero vectors is a semantic argument and I did not takes points off for including or not including $b = 0$ as an answer. Otherwise, $b = 2, -2$.

4. The diagonal of the unit cube is $a = (1, 1, 1)$ and $b = (1, 1, 0)$ is a diagonal of one of its faces. If θ is the angle between the two diagonals then $a \cdot b = 2 = |a||b|\cos(\theta) = \sqrt{3}\sqrt{2}\cos(\theta)$. So, $\theta = \arccos(\frac{2}{\sqrt{6}}) = \arccos(\sqrt{\frac{2}{3}})$.

5.

$$\text{proj}_a b = a(a \cdot b)/|a|^2 = -\frac{4}{25}(3, 0, 4)$$

and

$$\text{comp}_a b = |\text{proj}_a b| = \frac{4}{5}.$$

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Given that the book gives a slightly different definition of the scalar projection, I also accepted $\text{comp}_a b = -\frac{4}{5}$.

Part 3

1.

$$(-3,2,2)\text{x}(6,3,1) = \begin{vmatrix} i & j & k \\ -3 & 2 & 2 \\ 6 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} i - \begin{vmatrix} -3 & 2 \\ 6 & 1 \end{vmatrix} j + \begin{vmatrix} -3 & 2 \\ 6 & 3 \end{vmatrix} k = -4i + 15j - 21k.$$

The vector pointing in this direction and of unit length is $\frac{1}{\sqrt{682}}(-4, 15, -21)$.

2. First we calculate the normal to the plane. The normal will be perpendicular to $(2, -4, 6) - (1, 1, 1) = (1, -5, 4)$ and $(5, 1, 3) - (1, 1, 1) = (4, 0, 2)$ so $n = (1, -5, 4)\text{X}(4, 0, 2) = (-5(2) + 4(0))i - (1(2) - 4(4))j + (1(0) - (-5)4)k = -10i + 14j + 20k$. Thus the equation for the plane is $-10(x - 1) + 14(y - 1) + 20(z - 1) = 0$ or $-5x + 7y + 10z = 14$.

3.The equations for the planes are

$$\begin{aligned} x + y + z &= 1 \\ x + z &= 0 \end{aligned}$$

Subtracting the second equation from the first we see that $y = 1$. That combined with second equation, $x = -z$, gives you the parametric form form for the line by letting $x = t$ we get

$$\begin{aligned} x &= t \\ y &= 1 \\ z &= -t \end{aligned}$$

4. Let L be the given line. Then $(1, 1, 0)$ is the point on L corresponding to $t = 0$. L is in the direction of $a = (1, -1, 2)$ and $b = (1, 0, 2)$ is the vector joining $(1, 1, 0)$ and $(0, 1, 2)$. Then

$$b - \text{proj}_a b = (-1, 0, 2) - \frac{(1, -1, 2) \cdot (-1, 0, 2)}{1^2 + (-1)^2 + 2^2}(-1, -1, 2) = (-\frac{3}{2}, \frac{1}{2}, 1)$$

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is a direction vector for the required line. So one parametric form for the line is

$$
\begin{aligned}
x &= -\frac{3}{2}t \\
y &= 1 + \frac{1}{2}t \\
z &= 2 + t
\end{aligned}
$$

5. By equation 8 in the book on page 682, the distance is

$$
D = \frac{1}{\sqrt{16 + 36 + 1}}[4(3) + (-6)(-2) + 1(7) - 5] = \frac{26}{\sqrt{53}}
$$