This is part 2 (of 3) of the weekly homework. It is due Aug 13 in the mailbox of Jon. More problems to this lecture can be found on pages 970-971 and 976-977 in the book.

SUMMARY.

- $\text{curl}(P,Q,R) = (R_y - Q_z, P_z - R_x, Q_x - P_y)$ the **curl** of $F = (P,Q,R)$.
- The curl of a gradient field vanishes: $\text{curl}\nabla f = (0,0,0)$.
- $D : (u,v) \mapsto r(u,v) = (x(u,v), y(u,v), z(u,v))$ **surface** $S = r(D)$.
- $\int\int_S f\, dS = \int_D f(r(u,v))|r_u \times r_v|\, dA$ **surface integral** (generalizes area integral).
- $\int\int_S F \cdot dS = \int_D F(r(u,v)) \cdot (r_u \times r_v)\, dA$ **flux integral**.
- $\boxed{\int\int_S \text{curl}(F) \cdot dS = \int_C F \cdot dr}$ **Stokes theorem**, C: boundary of S.

1) (4 points) Evaluate $\int\int_S yz\, dS$, where S is the surface with parametric equation $x = uv, y = u+v, z = u-v, u^2+v^2 \leq 1$.

2) (4 points) Evaluate the flux integral $\int\int_S \text{curl}(F) \cdot dS$ for $F(x,y,z) = (xy, yz, zx)$, where S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $[0,1] \times [0,1]$ and has upward orientation.

3) (4 points) Evaluate the same flux integral as in the previous question but using Stokes theorem.

4) (4 points, compare problem 12a in 13.7) Use Stokes theorem to evaluate $\int_C F \cdot dr$, where $F(x,y,z) = (x^2y, x^3/3, xy)$ and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$, oriented counterclockwise as viewed from above.

5) (4 points) If S is the surface $x^6 + y^6 + z^6 = 1$ and F is a smooth vector field, show that $\int\int_S \text{curl}(F) \cdot dS = 0$.

CHALLENGE PROBLEM:

Use Stokes theorem to show that $\int_C (f\nabla g + g\nabla f) \cdot dr = 0$ for any closed curve C in space and any two functions f, g. (Hint: the identity also follows from the fundamental theorem of line integrals).

SUPER CHALLENGE PROBLEM:

Try to figure out, how Stokes theorem would look like in higher dimensions: in four dimensions, it is useful in special relativity.

Start: In dimension d, the curl is a field $\text{curl}(F)_{ij} = \partial_{x_j} F_i - \partial_{x_i} F_j$ with $\binom{d}{2}$ components. In 4 dimensions, it has 6 components. In d dimensions, a surface element in the $i-j$ plane is written as dS_{ij}. The flux integral of the curl of F through S is defined as $\int\int \text{curl}(F) \cdot dS$, where the dot product is $\sum_{i<j} \text{curl}(F)_{ij} dS_{ij}$. If S is given by a map r from a planar domain D to R^d, $U = \partial_u X$ and $V = \partial_v X$ are tangent vectors to that plane and $dS_{ij}(u,v) = (U_i V_j - U_j V_i)\, dudv$.