This is part 3 (of 3) of the weekly homework. It is due July 23 at the beginning of class. More problems to this lecture can be found on pages 818-820 and 827-829 in the book.

SUMMARY.

- $\nabla f(x,y) = (0,0)$ **critical point** or **stationary point** (candidate for max or min).
- $H(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$ **Hessian**. $D = \det(H(x,y)) = f_{xy}f_{yy} - f_{xy}^2$ **determinant**.
- $D > 0, f_{xx} < 0$ **local maximum**, $D > 0$, $f_{xx} > 0$ **local minimum**, $D < 0$ **saddle point**.
- Extremize $f(x,y)$ under the constraint $g(x,y) = c$: Solve $g = c, \nabla f(x,y) = \lambda \nabla g(x,y)$ with **Lagrange multiplier** λ, (3 equations for 3 unknowns x, y, λ). (In 3D: 4 eqns,4 variables).

1) (4 points, compare problems 5-14 in 11.7) Find all the extrema of the function $f(x,y) = xy^3 - yx^3$ and determine whether they are maxima, minima or saddle points.

2) (4 points, compare problems 23-28 in 11.7) Find the extrema of the function $f(x,y) = e^{-x^2-y^2}(x^2+2y^2)$ inside the disk $x^2+y^2 < 4$.

3) (4 points, compare problems 18-19 in 11.8) Find the extrema of the function $f(x,y) = e^{-x^2-y^2}(x^2+2y^2)$ on the circle $g(x,y) = x^2+y^2 = 4$ using the method of Lagrange multipliers. What is the maximum of $f(x,y)$ inside the disk $x^2+y^2 \leq 4$?

4) (4 points, compare problem 6 in Reviews) Find the points on the surface $xy^2 - z^3 = 2$ that are closest to the origin.

5) (4 points) Let a, b, c be non-negative constants and let F be the function $F(x,y,z) = -x\log(x) - y\log(y) - z\log(z) - ax - by - cz$. Find the stationary points of F, where $x > 0, y > 0, z > 0$ and $x + y + z = 1$.

Remark: The last problem appears in thermodynamics and is relevant to biology or chemistry. If x, y, z are the probabilities that a system is in state X, Y, Z and a, b, c are the energies for these states. Then $-x\log(x) - y\log(y) - z\log(z)$ is called the **entropy** of the system and $E = ax + by + cz$ is the **energy**. The number $F(x,y,z)$ is called the **free energy**. If energy is fixed, nature tries to maximize entropy. Otherwise it tries to **minimize the free energy** $F = S - E$. If we extremize F under the constraint of having total probability $G(x,y,z) = x + y + z = 1$, we obtain the so called **Gibbs distribution**. In this problem you find this distribution.

CHALLENGE PROBLEM:

What does it mean that the Lagrange multiplier λ is zero in a constrained optimization problem?

SUPER CHALLENGE PROBLEM:

How would a classification of critical points and formulate the criterium for local maxima or local minima for functions $f(x,y,z)$ of three variables?