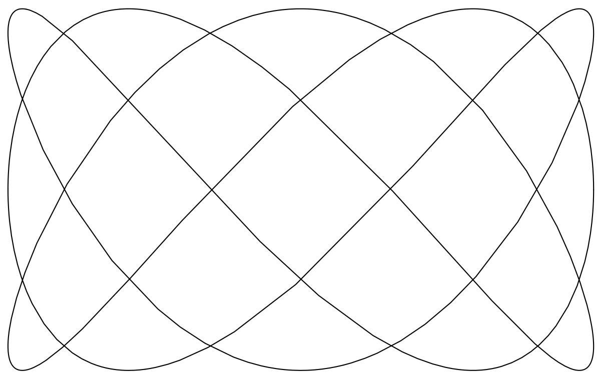
Problem 1) TF questions (50 points)

In each of the 25 questions, * denotes the place with the correct answer.

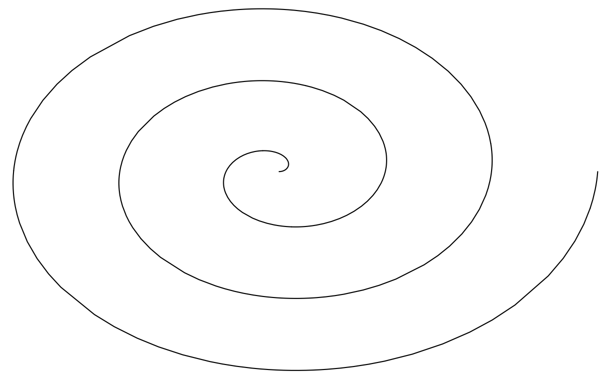
*	F	A nonzero velocity vector $r'(t)$ of $r(t)$ is tangent to the curve at $r(t)$.
*	F	The speed of the plane curve $r(t) = (\sin(t), t)$ at $t = 0$ is $\sqrt{2}$.
T	*	The acceleration vector can never be parallel to the velocity vector.
*	F	If all directional derivatives $D_u f(\vec{x})$ at a point $\vec{x}$ vanish then $\vec{x}$ is a critical point: $\nabla f = 0$.
*	F	The directional derivative $D_u(f)$ at a point $\vec{x} = (x, y, z)$ vanishes if u is a vector tangent to the surface $f = c$.
*	F	$\int_a^b \int_c^d 1 \, dxdy = (d - c)(b - a)$.
T	*	$(0, 1)$ is a critical point of $f(x, y) = x^2 y - 4xy + y^3/3$.
*	F	The functions $f(x, y)$ and $g(x, y) = 4f(x, y) + 3$ have the same critical points.
*	F	$d/dt \sin(x(t)y(t)) = \cos(x(t)y(t))(y(t)x'(t) + x(t)y'(t))$.
*	F	If a function $f(x, y) = ax + by$ has a critical point, then f is identically zero.
*	F	The point $(0, 0)$ is a local maximum of the function $f(x, y) = -x^2 - y^2$.
T	*	If (x, y) is a local maximum of $f(x, y)$ constrained to $g(x, y) = c$, then the gradient of f vanishes.
*	F	$f_{xxyxyx} = f_{yyxxxx}$ for $f(x, y) = \exp(\sin(\cos(y + x^{14}) + \exp(x)))$.
*	F	$\nabla f(x, y) + \nabla g(x, y) = \nabla(f + g)(x, y)$.
*	F	If $(1, 1)$ is a critical point of f and $D_u D_u f(1, 1) > 0$ for all vectors u, then $(1, 1)$ is a local minimum.
*	F	Assume (x_0, y_0) is a critical point of $f(x, y)$ and $f_{xx} = 0$, $f_{xy} \neq 0$, then (x_0, y_0) is a saddle point.
T	*	The function $x^4 - y^4$ has a critical point at $(0, 0)$ which is a local maximum.
*	F	It is possible that some directional derivative $D_u(x, y)$ is zero even though the gradient $\nabla f(x, y)$ is nonzero.
T	*	A function $f(x, y)$ on the closed disc $x^2 + y^2 \leq 1$ always has the maximum at $x^2 + y^2 < 1$ (away from the boundary).
*	F	The linear approximation of the function $f(x, y) = 3x + 5y$ satisfies $L(x, y) = f(x, y)$.
*	F	The maximum of a function $f(x, y)$ on the square $[0, 1] \times [0, 1]$ is always bigger or equal then the average value $\int_0^1 \int_0^1 f(x, y) \, dydx$.
*	F	If $f(x, y) = \sin(x) + \sin(y)$, then $-\sqrt{2} \leq D_u f(x, y) \leq \sqrt{2}$.
*	F	If the directional derivative $D_u(f)(x_0, y_0)$ is independent of u, then (x_0, y_0) is a critical point.
*	F	A maximum of $f(x, y)$ is also a maximum, when we constrain it to a curve $g(x, y) = c$.
T	*	If $f(x, y)$ has two local maxima on the plane, then f must have a local minimum on the plane.

Problem 2) Curves (20 points)

Match the equations with the curves and justify briefly your choice.



I



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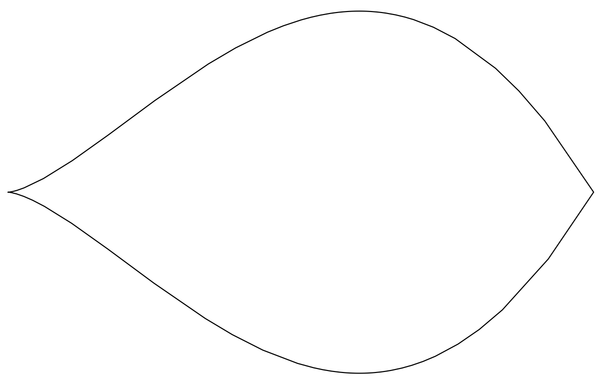

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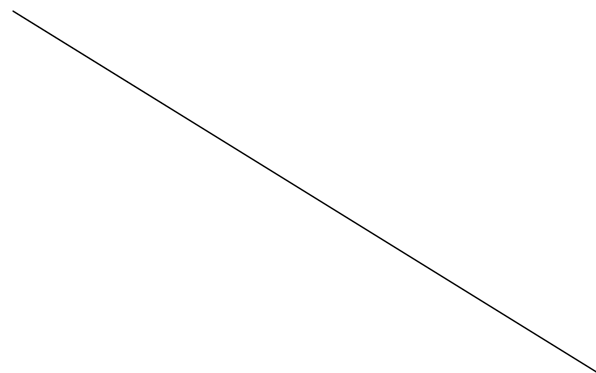
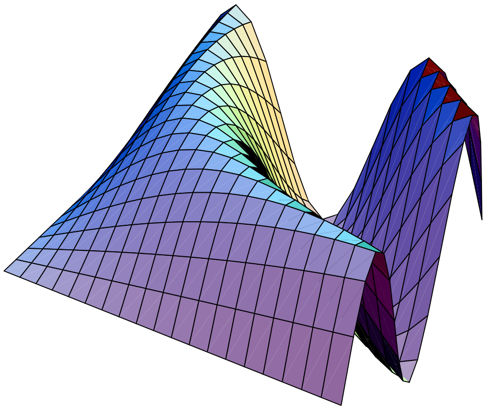


IV

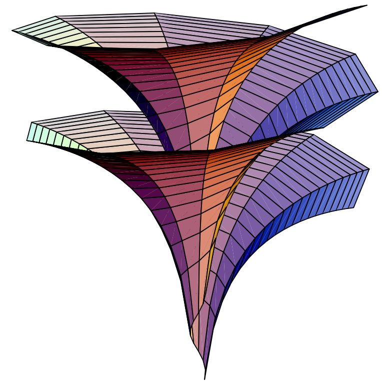
Enter I,II,III,IV here	Equation	Short Justification
I	$r(t) = (\cos(3t), \sin(5t))$	This is a Lissajoux figure, closed $r(2\pi) = r(0)$ no corners.
II	$r(t) = (t\cos(3t), t\sin(3t))$	$(\cos(3t), \sin(3t))$ would be a circle, have a time dependent radius $r = t$.
III	$r(t) = (t^2, t^3 - t^5)$	For $t = -1, t = 0, t = 1$ have $y(t) = 0$. For $t = -1$ and $t = 1$ are at $(1, 0)$ but velocity vectors differ (this leads to a corner)
IV	$r(t) = (\cos(t)^2, \sin(t)^2)$	$x(t) = \cos(t)^2, y(t) = \sin(t)^2 = 1 - x(t)$, therefore $x(t) + y(t) = 1$, this is a line.

Problem 3) Parametric Surfaces (20 points)

Match the parametric surfaces with their equations.



I



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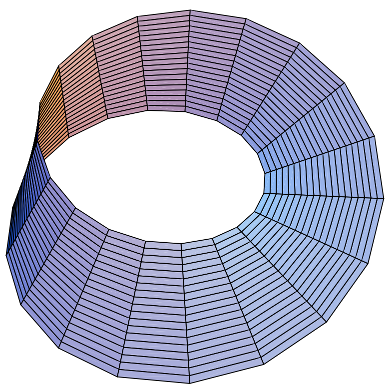
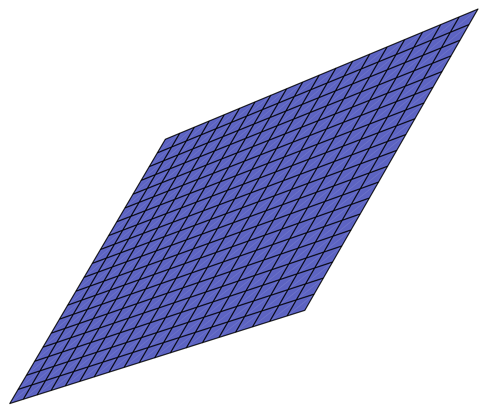


III



IV

Enter I,II,III,IV here	Equation
IV	$(u,v) \mapsto (u,v,u+v)$
I	$(u,v) \mapsto (u,v,\sin(uv))$
III	$(u,v) \mapsto ((2+v\cos(\pi u))\cos(2\pi u),(2+v\cos(\pi u))\sin(2\pi u),v\sin(\pi u))$
II	$(u,v) \mapsto (\cos(u)\sin(v),\sin(u)\sin(v),\cos(v)+\log(\tan(v/2))+u/5)$

Problem 4) Partial derivatives (20 points)

a) Verify that $f(x,y,z) = 3x^2z - 3yz - z^3 + xy$ solves the Laplace equation $f_{xx} + f_{yy} + f_{zz} = 0$.

Solution: $f_{xx}(x,y,z) = 6z$, $f_{yy} = 0, f_{zz} = -6z$. $f_{xx} + f_{yy} + f_{zz} = 6z - 6z = 0$.

b) Verify that if $f(x,y,z) = 1$ is an ellipsoid centered at zero (with x-intercepts $\pm a$, y-intercepts $\pm b$ and z-intercepts $\pm c$), then $xf_x + yf_y + zf_z = 2f$.

Solution: $f(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

$f_x(x,y,z) = 2x/a^2, f_y(x,y,z) = 2y/b^2, f_z(x,y,z) = f_z/c^2$. $f_xx + f_yy + f_zz = 2x^2/a^2 + 2y^2/b^2 + 2z^2/c^2 = wf$.

Problem 5) Tangent planes (20 points)

a) Find the equation for the tangent line to the level curve $2x^2+3y^2+xy^5 = 6$ at the point $(1,1)$.

Solution: $\nabla f(x,y) = 4x + y^5, 6y + 5xy^4)$ and $\nabla f(1,1) = (5,11)$ The tangent has line has is normal to this vector $(5,11)$ so that $5x+11y = d$. Because $(1,1)$ is on the line, $d = 5+11 = 16$. The solution is $5x + 11y = 16$.

b) Find the equation for the tangent plane to the surface $f(x,y,z) = zx^2 - y^3 - z^3xy = 1$ at the point $(2,1,1)$.

Solution: the normal to the plane is $\nabla f(2,1,1) = (3,-5,-2)$ so that the plane is $3x-5y-2z = d$. The constant d is obtained by plugging in $(2,1,1)$: we have $3x - 5y - 2z = -1$.

Problem 6) Linear approximation (20 points)

a) Find the linear approximation $L(x,y,z)$ of $f(x,y,z) = x^3 + y^3 - 3xyz$ at the point $(1,1,1)$.

Solution: $f(x,y,z) = 2x^2+3y^2+xy^5$. $\nabla f(x,y,z) = (3x^2-3yz, 3y^2-3xz, -3xy)$. $\nabla f(1,1,1) = (0,0,3)$, $L(x,y,z) = f(1,1,1) + \nabla f(1,1,1) \cdot (x-1, y-1, z-1) = -1 - 3(z-1) = 2 - 3z$.

b) Estimate $f(1.01, 0.99, 1.02)$ for the same function f.

Solution: $f(1,1,1) = -1$, $f(1.01, 0.99, 1.02) \sim L(1.01, 0.99, 1.02) = -1 + 3(1.02 - 1) = -1.06$.

Problem 7) Extrema (30 points)

a) Classify all critical points of $f(x,y) = x^3/3 - x - (y^3/3 - y)$.

Solution: $\nabla f(x,y) = (x^2 - 1, -(y^2 - 1)) = (0,0)$ so that the critical points are $(1,1), (-1,-1), (1,-1), (-1,1)$. The Hessian is $H(x,y) = \begin{bmatrix} 2x & 0 \\ 0 & -2y \end{bmatrix}$ and has determinant $D = -4xy$ and $f_{xx} = 2x$.

Point	Det(H)	f_{xx}	type
$(1,1)$	$D = -4$	$f_{xx} = 2$	saddle
$(-1,-1)$	$D = -4$	$f_{xx} = -2$	saddle
$(1,-1)$	$D = 4$	$f_{xx} = 2$	min
$(-1,1)$	$D = 4$	$f_{xx} = -2$	max

b) A wooden drawer of length x, width y and height z is a rectangular box which is open on the top. It should have the volume 32 cubic feet. For which dimensions are the material costs minimal?

Solution: To minimize $f(x,y,z) = xy + 2yz + 2xz$ under the constraint $g(x,y,z) = xyz = 32$ we solve Lagrange equations

$$\begin{aligned} y + 2z &= \lambda yz \\ x + 2z &= \lambda xz \\ 2y + 2x &= \lambda xy \\ xyz &= 32 \end{aligned}$$

Subtracting 2) from the 1) gives $(y - x) = \lambda(y - x)z$. If $y - x$ is not zero, then $\lambda z = 1$ which in 1) would give $y + 2z = y$ or $z = 0$ contradicting 4). Therefore, $y - x = 0$ or $y = x$. Equation 3) gives $4x = \lambda x^2$. Again $x = 0$ would contradict the 4) so that we can divide by x and get $4 = \lambda x$. Equation 2) gives $x + 2z = 4z$ or $x = 2z$. We have now $x = y = 2z$. Using 4) gives $xyz = 4z^3 = 32$ so that $z = 2$ and $x = 4$, $y = 4$.

Problem 8) Integrals (20 points)

a) Find the area of the region D enclosed by the lines $x = \pm 2$ and the parabolas $y = 1 + x^2$, $y = -1 - x^2$.

Solution: $\int_{-2}^{2} \int_{-(1+x^2)}^{1+x^2} 1 \; dxdy = \int_{-2}^{2} (1 - x^2) dx = x - x^3/3|_{-2}^{2} = 56/3$

b) Find the integral of $f(x,y) = x^2$ on the same region as in a). (The result can be interpreted as a **moment of inertia**).

Solution: $\int_{-2}^{2} \int_{-(1+x^2)}^{1+x^2} x^2 \; dxdy = \int_{-2}^{2} x^2(1 - x^2) dx = x^3/3 - x^5/5|_{-2}^{2} = 544/15$