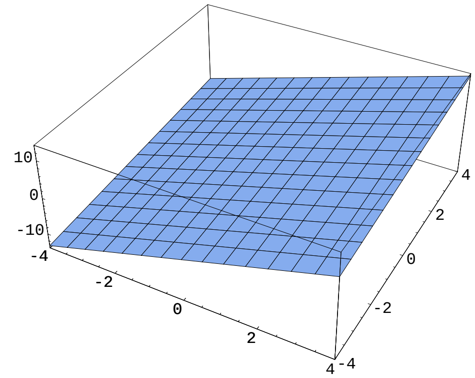
Problem 1) TF questions (50 points)

In each of the 25 questions, * denotes the place with the correct answer. No justifications were needed.

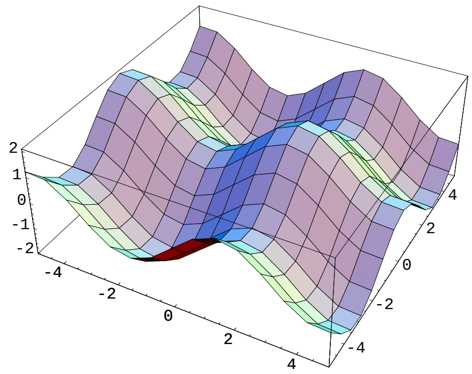
T	*	The vectors $(3, -2, 1)$ and $(-6, 4, 2)$ are parallel.
T	*	The length of the vector $(3, 4, 0)$ is 25.
*	F	For any two vectors, $v \cdot w = w \cdot v$.
T	*	For any two vectors, $v \times w = w \times v$.
*	F	$\|(u \times v) \cdot w\| = \|(u \times w) \cdot v\|$.
*	F	The vectors $(1, 1)$ and $(1, -1)$ are orthogonal.
*	F	For any vector v one has $v \times (2v) = 0$.
T	*	If we add the vector $(1, 1, 1)$ to the point $P = (2, 3, 4)$, we obtain the point $Q = (1, 2, 3)$.
*	F	The surface $x^2 - y^2 + z^2 = 1$ is a one-sheeted hyperboloid.
*	F	The set of points which have distance 1 from a line is a cylinder.
T	*	$\|v \times w\| = 0$ implies $v = 0$ or $w = 0$.
T	*	The intersection of two planes is always a line.
*	F	$x^2 + 2x + y^2 + z^2 = 0$ is a sphere.
*	F	If $u + v$ and $u - v$ are orthogonal, then u and v have the same length.
*	F	If P, Q, R are 3 different points that don't lie in a line, then $(P - Q) \times (Q - R)$ is a vector orthogonal to the plane.
T	*	Two lines in three dimensional space always intersect in a point.
*	F	The line $r(t) = (1 + 2t, 1 + 3t, 1 + 4t)$ intersects the plane $2x + 3y + 4z = 9$ at a right angle.
T	*	If in rectangular coordinates, a point is given by $(1, 1, 0)$, then its spherical coordinates are $(\rho, \theta, \phi) = (\sqrt{2}, \pi/2, \pi/2)$.
*	F	If the velocity vector of the curve $r(t)$ is a nonzero constant vector v for all times t, then the curve is a straight line.
T	*	Every point on the sphere of radius ρ is determined alone by its angle ϕ from the z axes.
T	*	The equation $r = 3$ in cylindrical coordinates is a sphere.
T	*	The set of points which satisfy $x^2 - 2y^2 - 3z^2 = 0$ are on an ellipsoid.
*	F	A surface which is given as $r = 2 + \sin(z)$ in cylindrical coordinates stays the same when we rotating it around the z axes.
*	F	The identity $\|v \cdot w\|^2 + \|v \times w\|^2 = \|v\|^2\|w\|^2$ holds for all vectors v, w.
T	*	If $v \times w = (0, 0, 0)$, then $v = w$.

Problem 2) Functions of two variables (20 points)

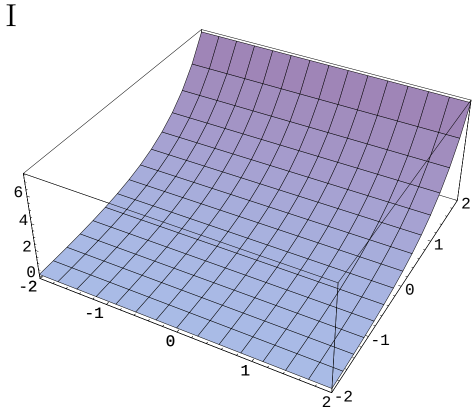
Match the equation with their graphs and justify briefly your choice.



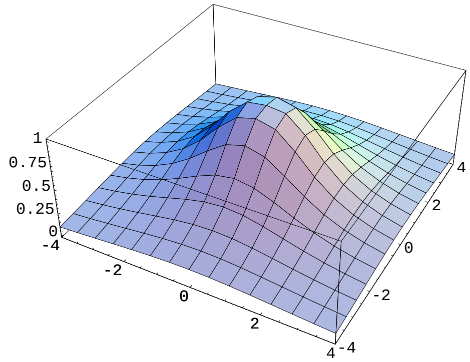
I



II



III

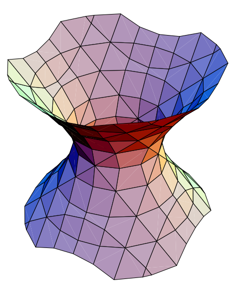


IV

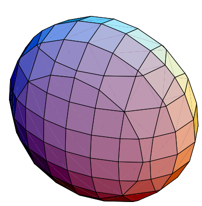
Enter I,II,III,IV here	Equation	Short justification
IV	$z = \sin(\pi/(2 + x^2 + y^2))$	decay at infinity
II	$z = \sin(x) + \cos(y)$	waves
III	$z = e^y$	x-independence
I	$8x + 2y + 3z = 0$	linear

Problem 3) Quadrics (30 points)

Match the equation with their graphs and justify briefly your choice.



I



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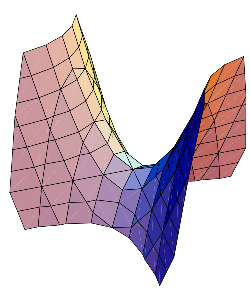
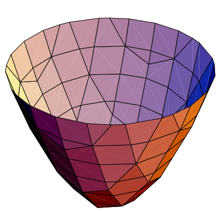
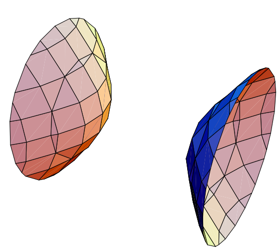


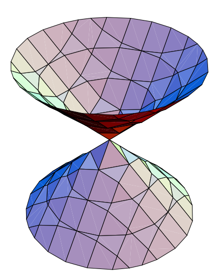
III



IV



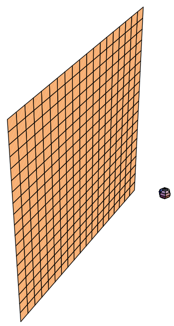
V



VI

Enter I,II,III,IV,V,VI here	Equation	Short justification
IV	$x^2+y^2-z=1$	$z=1+r^2$ parabola
II	$x^2+2y^2+z^2=1$	x,y,z are all ≤ 1
III	$x^2-y^2-z=1$	traces: parabolas and hyperbolas
VI	$x^2+y^2=z^2$	rotate $z^2=r^2$ around z axes
V	$x^2-y^2-z^2=1$	$x=0$ has no solution
I	$x^2+y^2-z^2=1$	$z=0$ has circle as solution

Problem 4) Distances (20 points)



Given the vectors $v=(1,1,0)$ and $w=(0,0,1)$ and the point $P=(2,4,-2)$. Let Σ be the plane which goes through the origin and contains the vectors v and w.

a) Determine the distance from P to the origin.

$2\sqrt{6}$

$\sqrt{4+16+4}=\sqrt{24}=2\sqrt{6}$.

b) Determine the distance from P to the plane Σ.

$\sqrt{2}$

Solution: $\vec{n}=(1,1,0)\times(0,0,1)=(1,-1,0)$ is the normal vector to the plane. The distance is the scalar projection of $PO=(2,4,-2)-(0,0,0)$ onto this vector:

$$|\vec{n}\cdot PO|/|\vec{n}|=|(2,4,-2)\cdot(1,-1,0)|/\sqrt{2}=2/\sqrt{2}=\sqrt{2}\,.$$

Problem 5) Coordinate systems (20 points)

a) Let $P=(x,y,z)=(-1,1,-1)$ (rectangular coordinates). Find the spherical coordinates (ρ,θ,ϕ) of P.

$(\sqrt{3},\frac{3\pi}{4},\frac{\pi}{2}+\text{artan}(\frac{1}{\sqrt{2}}))$

Solution: ρ is the length of (x,y,z) which is $\sqrt{3}$. The vector θ is obtained by looking at the $x-y$ plane. The vector $(-1,1)$ points into the left upper quadrant. It forms the angle $3\pi/4$ with the x-axes. To obtain the vector ϕ, look at the plane which contains the z axes and the point P. The point is on the lower hemisphere: $\phi=\frac{\pi}{2}+\text{artan}(\frac{1}{\sqrt{2}})$.

b) Let $R=(r,\theta,z)=(4,\pi/3,-3)$ (cylindrical coordinates). Find the spherical coordinates of R.

$(5,\pi/3,\text{artan}(\frac{3}{4})+\frac{\pi}{2})$

Solution: The distance to the origin is $\sqrt{3^2+4^2}=5$ (look at a triangle in the plane which contains the z-axes and the point). The θ angle in spherical and cylindrical coordinates agree.

To get ϕ, look at the triangle with length $3,4,5$ which angle $\arctan(3/4)$. We are on the lower hemisphere, the angle ϕ between R and the z axis is $\pi/2+\arctan(3/4)$.

Problem 6) Various topics (20 points)

a) Find a parametric equation for the line through the points $(3,1,-1)$ and $(3,2,-6)$.

$r(t) = (3,1,-1) + t(0,1,-5)$

Solution. The vector $(0,1,-5)$ connects the two points.

b) Identify the surface, whose equation is given in cylindrical coordinates by $z = r^2$. Either name it or sketch the surface convincingly.

Paraboloid

Solution. In rectangular coordinates, the surface is $z = r^2 = x^2+y^2$. It is rotational symmetric and the xz and yz traces are parabolas $z = x^2$ or $z = y^2$.

Problem 7) Coordinate systems (20 points)

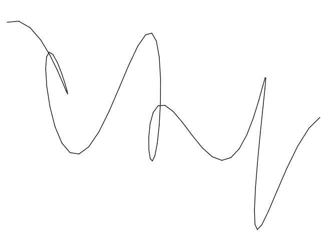
a) Identify the surface whose equation is given in spherical coordinates as $\theta = \pi/4$.

Halfplane: contains z axes as boundary and for example the point (1,1,0

b) Identify the surface whose equation is given in spherical coordinates as $\phi = \pi/4$.

Halfcone: rotational symmetric around z-axes, only $z \geq 0$ part of $x^2+y^2 = z^2$.

Problem 8) Curves (20 points)



Let $r(t)$ be the space curve $r(t) = (t^2, \sin(3\pi t), \cos(5\pi t))$.

a) Calculate the velocity vector of $r(t)$ at time $t = 1$.

$r'(t) = (2t, 3\pi\cos(3\pi t), 5\pi\sin(5\pi t)), r'(1) = (2, -3\pi, 0)$

b) Calculate the speed of $r(t)$ at time $t = 1$.

$\sqrt{4+9\pi^2}$