Problem 1) TF questions (50 points)

In each of the 25 questions, * denotes the place with the correct answer.

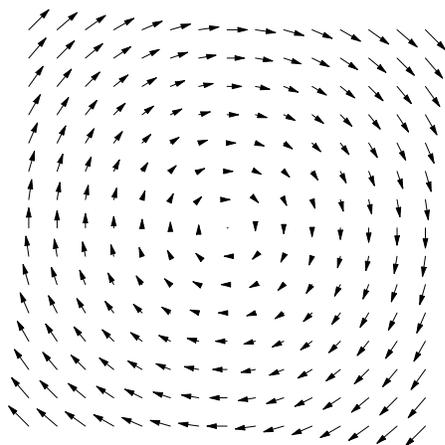| Answer		Statement
*	F	The scalar projection of a vector a onto a vector b is the length of the projection of a onto b.
*	F	If a, b are vectors, then $|a \times b|$ is a a scalar which gives the area of the parallelogram spanned by a and b.
*	F	The distance between two points A, B in space is the length of the curve $r(t) = A + t(B - A), t \in [0, 1]$.
T	*	$\int_a^b \int_c^d x \; dxdy = (d - c)(b^2 - a^2)/2$.
T	*	$(1, 1)$ is a critical point of $f(x, y) = xy - y^3/3$.
T	*	The surface $x^2 - y^2 + z^2 = -1$ is a one-sheeted hyperboloid.
T	*	$(u \times v) \cdot w = (u \times w) \cdot v$ for all vectors u, v, w.
*	F	The equation $r = 1$ in cylindrical coordinates is a cylinder.
*	F	The equation $\rho = z = \rho\cos(\phi)$ in spherical coordinates is half a cone.
T	*	The vector field $F(x, y, z) = (\sin(x), z, x)$ has zero divergence.
*	F	For any vector field F, the flux of the vectorfield curl(F) through the surface $x^4 + y^4 + z^4 = 1$ is zero.
T	*	If F is a vector field for which $\int_C F \cdot dr = 0$, where C is the unit circle, then F is a gradient field.
*	F	$\int_0^1 \int_0^1 xy \; dxdy = 1/4$.
*	F	For any numbers a, b satisfying $|a| \neq |b|$, the vector $(a - b, a + b)$ is orthogonal to $(a + b, b - a)$.
T	*	The line integral of $F(x, y) = (-y, x)$ along the positively oriented boundary of a region R is the area of R.
*	F	If F is a vector field tangent to a surface S, then the flux $\int \int_S F \cdot dS$ of F through S is zero.
*	F	The length of a curve does not depend on the chosen parameterization.
T	*	If F is a vector field in space, then curl$(2F) = 8$ curl(F).
*	F	A surface in space for which all normal vectors are parallel to each other must be a plane.
*	F	There are surfaces which bound a body of finite volume but which have infinite surface area.
T	*	There is no quadric for which both the parabola and the hyperbola appear as traces.
*	F	If $(u, v) \mapsto (r(u, v)$ is a surface, then $r_u(u, v) + r_v(u, v)$ is a vector which lies in the tangent plane at $r(u, v)$.
T	*	The flux of the curl of the vector field $F(x, y, z) = (x, y, z)$ through the ellipsoid $x^2 + 4y^2 + 9z^2 = 1$ is 1.
T	*	When changing from rectangular to spherical coordinates, one has to include an integration factor $\rho^2 \cos(\phi)$.
T	*	Let $F(x, y, z) = (z^2xy, x^2, z^3 + y)$. The flux of curl$(F)$ through the ellipsoid $x^2 + y^2 + 2z^2 = 1$ is the same as half the flux of curl(F) through the ellipsoid $x^2 + y^2 + 4z^2 = 1$.

Problem 2) Vector fields (20 points)

Note: in this problem, $\text{curl}(F) = 0$ means that $\text{curl}(F)(x,y) = 0$ vanishes for **all** (x,y). $\text{curl}(F) \neq 0$ means that $\text{curl}(F)(x,y)$ does not vanish for some (x,y). The same remark applies if curl is replaced by div.
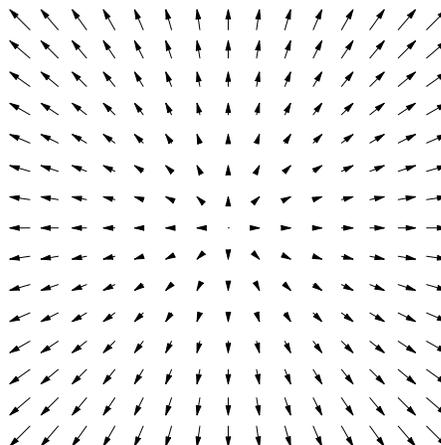
Circle the interior of the boxes to match the formulas of the vectorfields with the corresponding picture I,II,III or IV. Put also circles to indicate the vanishing or not vanishing of curl and div of the fields. In each of the four lines, you should finally have circled three boxes. No justifications are needed.

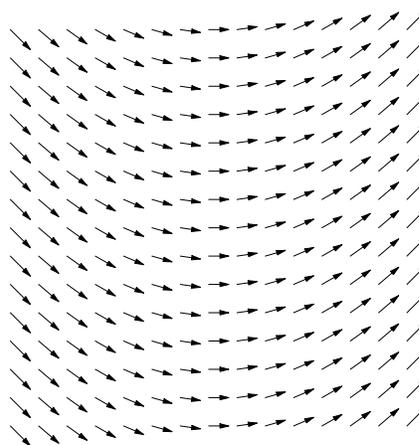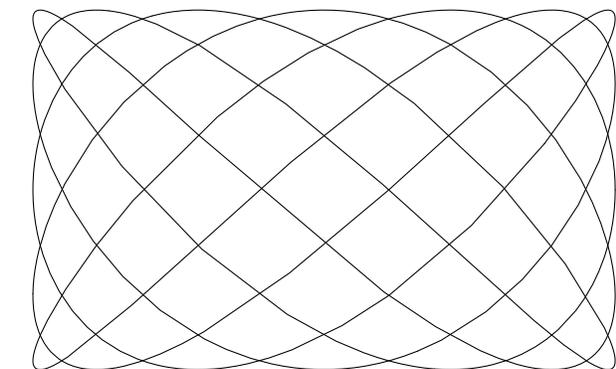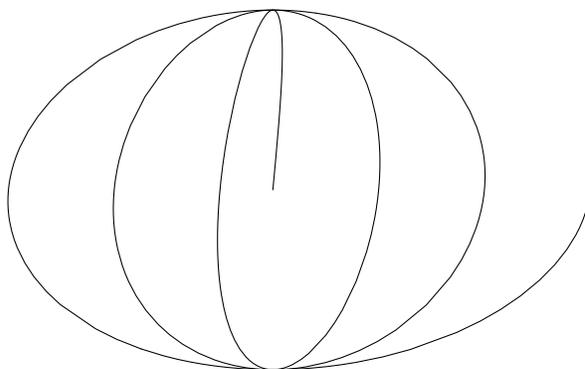| Vectorfield	I	II	III	IV	$\text{curl}(F) = 0$	$\text{curl}(F) \neq 0$	$\text{div}(F) = 0$	$\text{div}(F) \neq 0$
$F(x,y) = (0,5)$			O		O		O	
$F(x,y) = (y,-x)$	O					O	O	
$F(x,y) = (x,y)$		O			O			O
$F(x,y) = (2,x)$				O		O	O	

I

II

III

IV

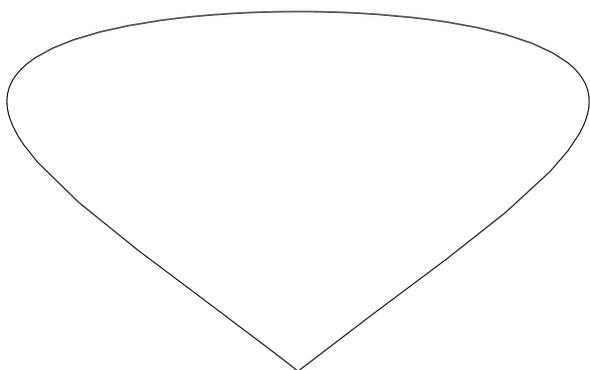Problem 3) Curves (20 points)

Match the equations with the curves. No justifications are needed.

I

II

III

IV
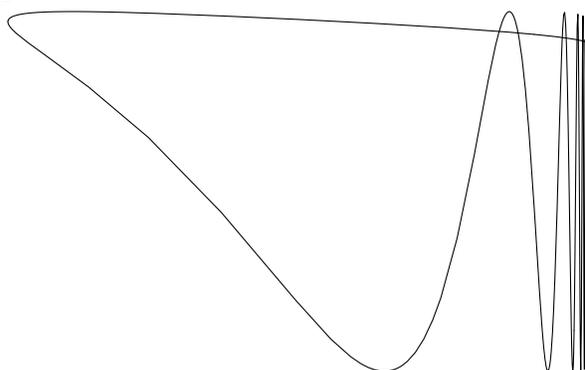
Enter I,II,III,IV here	Equation
III	$r(t) = (\sin(t), t(2\pi - t))$
I	$r(t) = (\cos(5t), \sin(7t))$
II	$r(t) = (t\cos(t), \sin(t))$
IV	$r(t) = (\cos(t), \sin(6/t))$

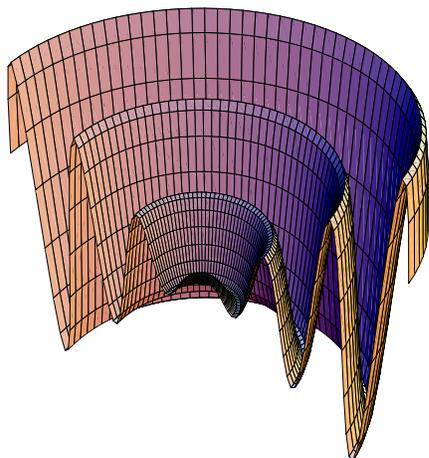## Problem 4) Parametric Surfaces (20 points)

Match the parametric surfaces with their equations. No justifications are needed.

I

II

III

IV
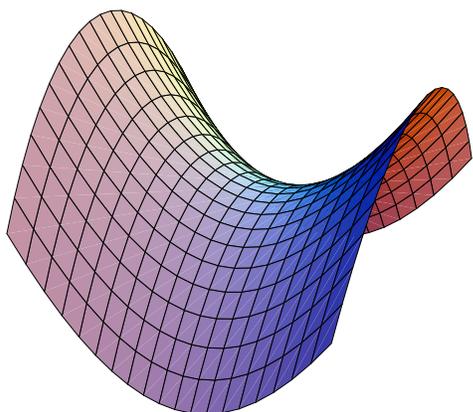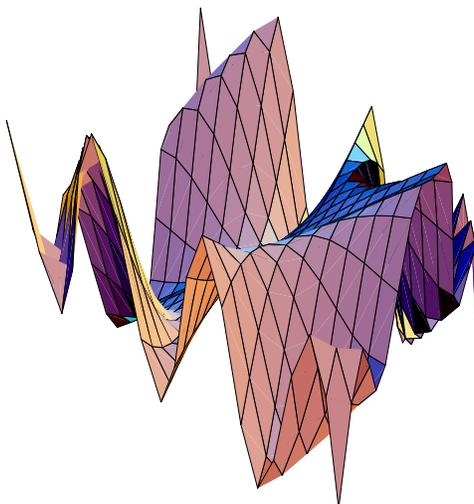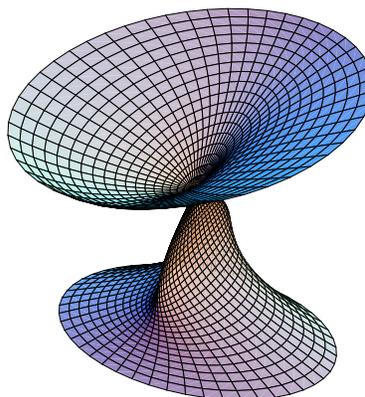
Enter I,II,III,IV here	Equation
III	$(u,v) \mapsto (u\cos(v), u\sin(v), u^2\cos(u)/(u+1))$
I	$(u,v) \mapsto (u, v, u^2 - v^2)$
II	$(u,v) \mapsto (u, v, v\sin(uv) + u\cos(v))$
IV	$(u,v) \mapsto ((u - \sin(u))\cos(v), (u - \cos(u))\sin(v), u)$

Problem 5) Quadrics (20 points)

Match the equation with their graphs. No justifications are needed.

I

II

III

IV
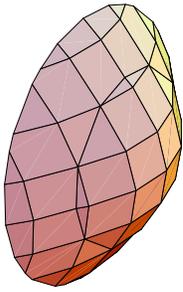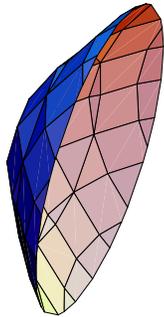
Enter I,II,III,IV here	Equation
I	$-x^2 + y^2 + z^2 = -1$
II	$x^2 - z^2 = y^2$
IV	$-x^2 + y^2 + z^2 = 1$
III	$x^2 + z^2 = y$

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Problem 6) Dot and Cross product (20 points)

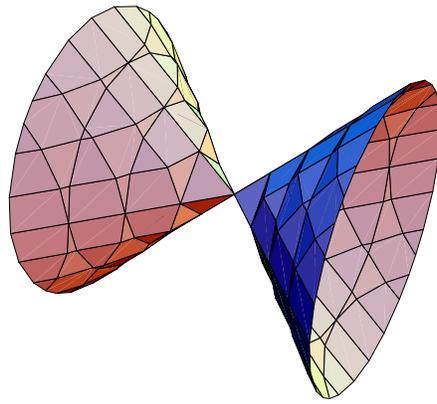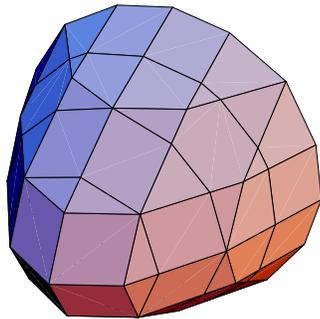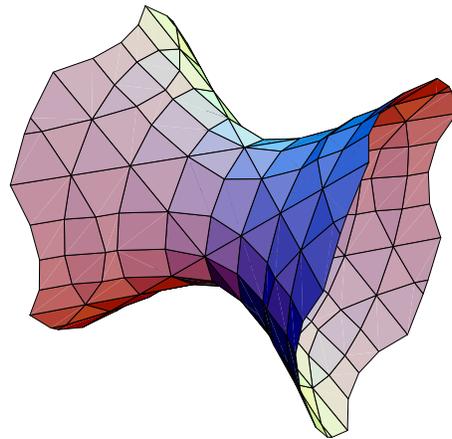a) Find the scalar projection of the vector $v = (3, 4, 5)$ onto the vector $(2, 2, 1)$.
Solution: $|(3, 4, 5) \cdot (2, 2, 1)|/|(2, 2, 1)| = \boxed{19/3}$.

b) Find the equation of a plane which contains the vectors $(1, 1, 0)$ and $(0, 1, 1)$ and contains the point $(0, 1, 0)$.

Solution: $(1, 1, 0) \times (0, 1, 1) = (1, -1, 1)$. The equation of the plane is $\boxed{x - y + z = -1}$.

Problem 7) Extrema (20 points)

Find all the local maxima, local minima and saddle points for the function $f(x, y) = y^3 - 3y + x^2$ and classify them.

Solution: $\nabla f(x, y) = (2x, 3y^2 - 3)$ vanishes for $x = 0, y = \pm 1$.
The Hessian matrix $H = \begin{bmatrix} 1 & 0 \\ 0 & 3y \end{bmatrix}$ has determinant $D = 3y$.

$(0, 1)$	$D = 3$	$H_{11} = 1$	minimum
$(0, -1)$	$D = -3$	$H_{11} = 1$	saddle point

Problem 8) Lagrange multipliers (20 points)

Find the points where the function $f(x, y) = x^2 y + 2$ takes on its maximum and minimum on the curve $x^2 + 2y^2 = 6$. You are required to solve this problem using Lagrange multipliers.

Solution: $\nabla f = (2xy, x^2)$, $\nabla g = (2x, 4y)$.
$2xy = \lambda 2x, x^2 = \lambda 4y, x^2 + 2y^2 = 6$. Now $y = \lambda$ and $x^2 = 4\lambda^2$. Plugging this into the third equation gives $4\lambda^2 + 2\lambda^2 = 6$. So, $\lambda = \pm 1$ and $y = \pm 1$ and $x^2 = 4$ so that we have to consider the points $(2, 1), (-2, 1), (2, -1), (2, 1)$. Evaluating f on these two points shows that $\boxed{(2, 1), (-2, 1) \text{ are maxima and } (2, -1), (2, 1) \text{ are minima.}}$

Problem 9) Double integrals (20 points)

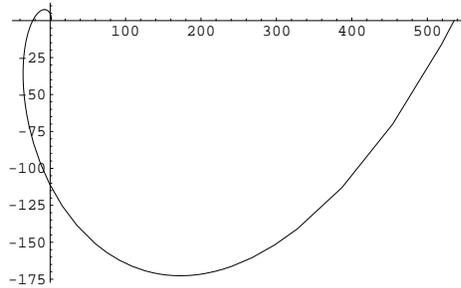Integrate $f(x, y) = x + 3y^2 + 1$ over the region $0 \le x \le 1$ and $0 \le y \le x$.
$\int_0^1 \int_0^x (x + 3y^2 + 1) \, dy dx = \int_0^1 (x^2 + x^3 + x) \, dx = 1/2 + 1/3 + 1/4 = \boxed{13/12}$.

Problem 10) Length of curve (20 points)

Sketch the plane curve $r(t) = (\cos(t)e^t, \sin(t)e^t)$ for $t \in [0, 2\pi]$ and find its length.

Solution:

$$\int_0^{2\pi} \sqrt{(\cos(t) - \sin(t))^2 e^{2t} + (\cos(t) + \sin(t))^2 e^{2t}} \; dt =$$
$$\sqrt{2}e^t|_0^{2\pi} = \boxed{\sqrt{2}(e^{2\pi} - 1)}.$$

Problem 11) Surface area (20 points)

Find the surface area of the surface $r(u, v) = (u, v, 2 + \frac{u^2}{2} + \frac{v^2}{2})$, where (u, v) is in the disc $u^2 + v^2 \le 1$.

Solution: $\int \int_D \sqrt{u^2 + v^2 + 1} \; dA = 2\pi \int_0^1 \sqrt{1 + r^2} r \; dr = 2\pi(1 + r^2)^{3/2}/3|_0^1 = \boxed{2\pi(\sqrt{8} - 1)/3}.$

Problem 12) Green's theorem (20 points)

a) Find the line integral of $F(x, y) = (xy, x)$ along the unit circle $C : t \mapsto r(t) = (\cos(t), \sin(t)), t \in [0, 2\pi]$ by calculating the actual line integral.

Solution: $\int_0^{2\pi}(\cos(t)\sin(t), \cos(t)) \cdot (-\sin(t), \cos(t)) \; dt = \int_0^{2\pi} \cos^2(t) + \sin^2(t)\cos(t) \; dt = \boxed{\pi}.$

b) Find the value of the line integral obtained in a) by evaluating a double integral.

Solution: The curl is $1 + x$. Integrating the curl over the disc is the integral $\int_0^1 \int_0^{2\pi}(1 + r\cos(\theta))r \; d\theta$, which is $\boxed{\pi}$.

Problem 13) Flux integral, divergence theorem (30 points)

a) Find the flux of the vector field $F(x, y, z) = (0, 0, x + z)$ through the sphere $x^2 + y^2 + z^2 = 9$, where the sphere is oriented with the normal pointing outside.

Solution: $\int_0^{2\pi} \int_0^\pi 9\cos(\phi)\sin(\phi) \; d\phi d\theta = \boxed{36\pi}.$

b) Find the flux of the vector field $F(x, y, z) = (0, 0, x + z)$ through the sphere $x^2 + y^2 + z^2 = 9$ with the divergence theorem.

Solution: the divergence is 1. The average of the divergence over the sphere of radius R is $4\pi R^3/3 = \boxed{36\pi}.$

c) Explain **in words** without doing any flux integral calculation nor invoking Stokes theorem,

why the flux integral of the vector field $F(x, y, z) = (0, 0, x + z)$ through any sphere with positive radius centered at $(0, 0, 0)$ is positive. A one or two sentence explanation is sufficient, but it should be formulated so that it makes sense also to somebody who does not know any calculus at all. Hint: Split up F as a sum $F = (0, 0, x) + (0, 0, z)$ and look at the two fluxes separately.

Solution:

The flux integral of $F(x, y, z) = (0, 0, x)$ is zero by symmetry (same flux on upper and lower hemisphere with opposit sign).
The flux integral of $F(x, y, z) = (0, 0, z)$ is larger on the upper then on the lower hemisphere.