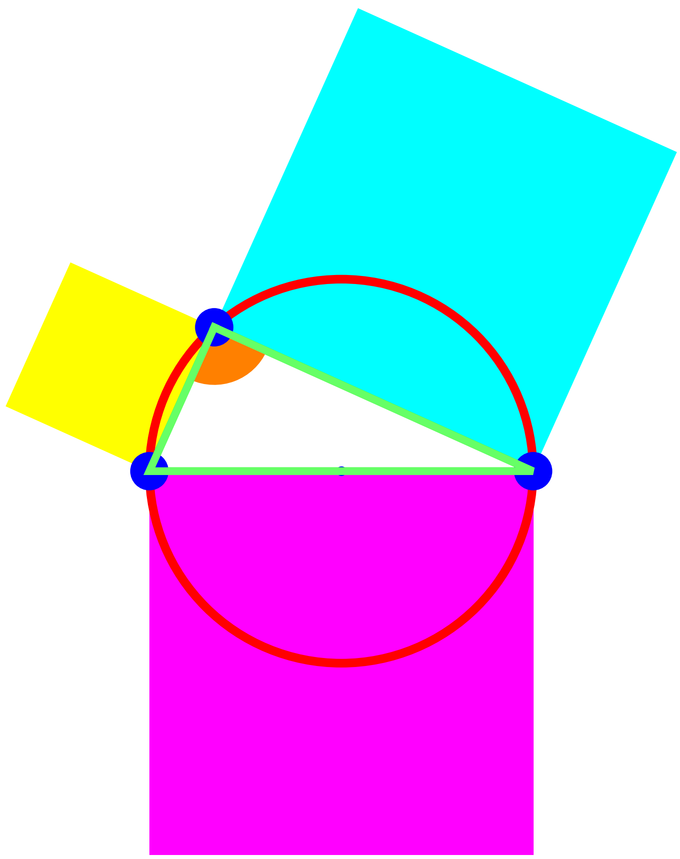
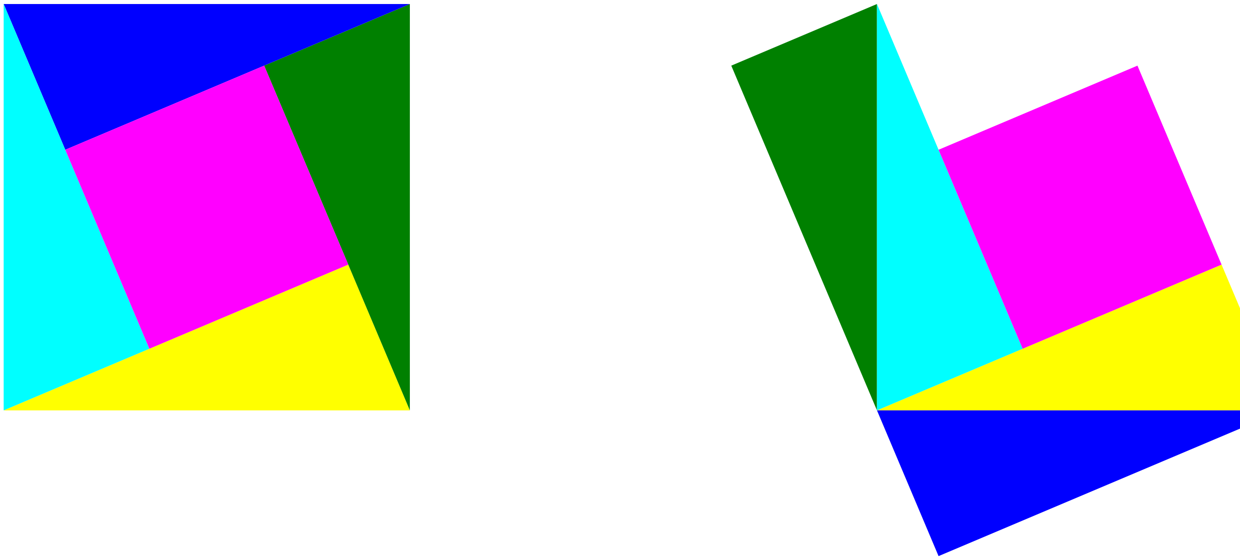
Lecture 3: Worksheets

Pythagoras theorem

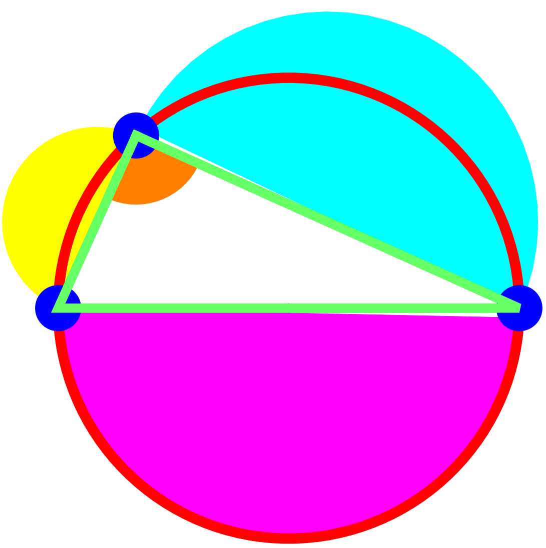
For all right angle triangles of side length a, b, c, the quantity $a^2 + b^2 - c^2$ is zero.



Here is a rearrangement proof:



a) Replace the squares above each edge of the triangle with a half disk. What is the relation between the half spheres?

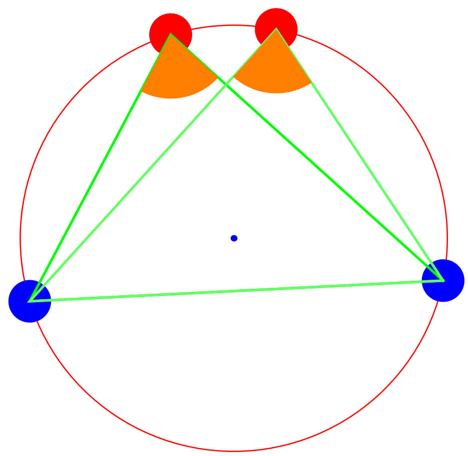


b) Lets look at a **3D Pythagoras theorem:** cut a corner off from a cube. Then the area square of the base is the sum of the area squares of the other triangles. This is called the **Faulhaber extension** of the Pythagoras theorem or the **de Gua theorem**.

Is there an elementary way to see that? We can compute it brute force by computing the area of the triangle $A = (a, 0, 0), B = (0, b, 0), C = (0, 0, c)$ which is $|\langle a, -b, 0\rangle \times \langle 0, b, -c\rangle|/2 = |\langle bc, ac, ab\rangle| = (bc^2 + ac^2 + ab^2)/4$.

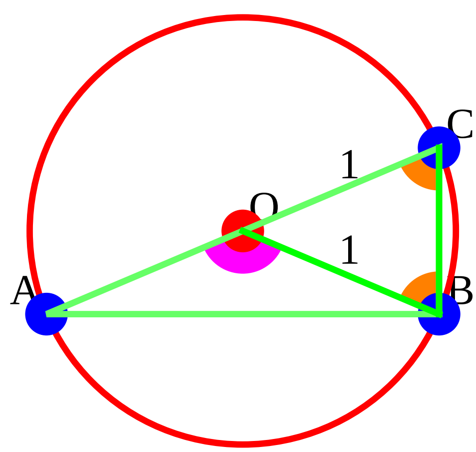
Thales theorem

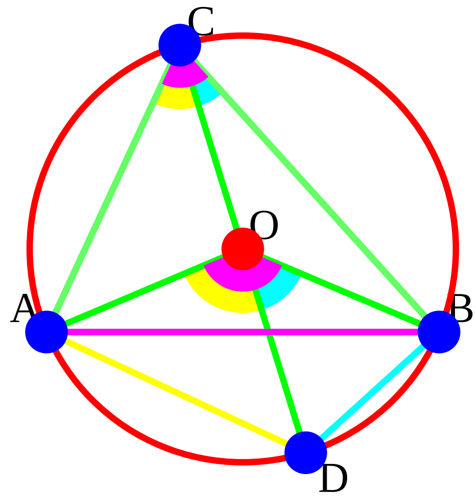
Thales of Miletus (625 BC -546 BC) showed that if a triangle inscribed in a fixed circle is deformed by moving one of its points on the circle, then the angle at this point does not change. Thales is considered the first modern Mathematician, his theorem is a prototype of a stability result.



Lets look first at the case when one side of the triangle goes through the center.
a) The triangle BCO is an isosceles triangle.

b) The central angle AOB is twice the angle ACB.





c) What is the relation between the angles AOD and ACD?

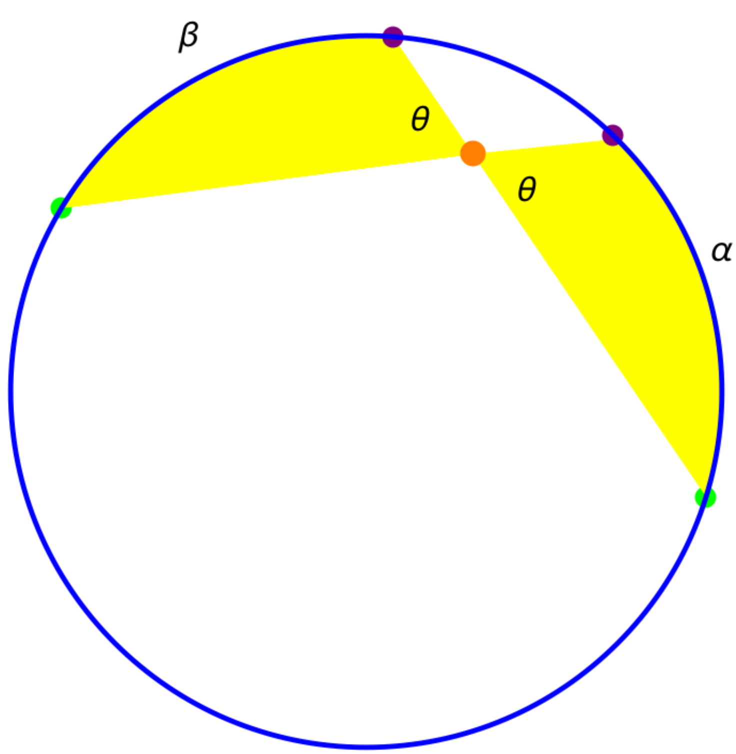
d) What is the relation between the angles DOB and DCB?

e) Find a relation between the central angle AOB and the angle ACB?

f) Why does the angle ACB not change if C moves on the circle?

A challenge

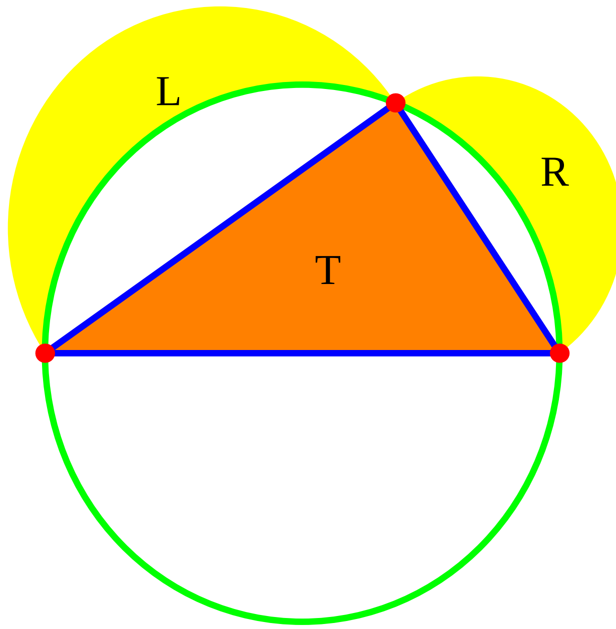
Can you express the sum of the arc lengths $\alpha + \beta$ in terms of θ?



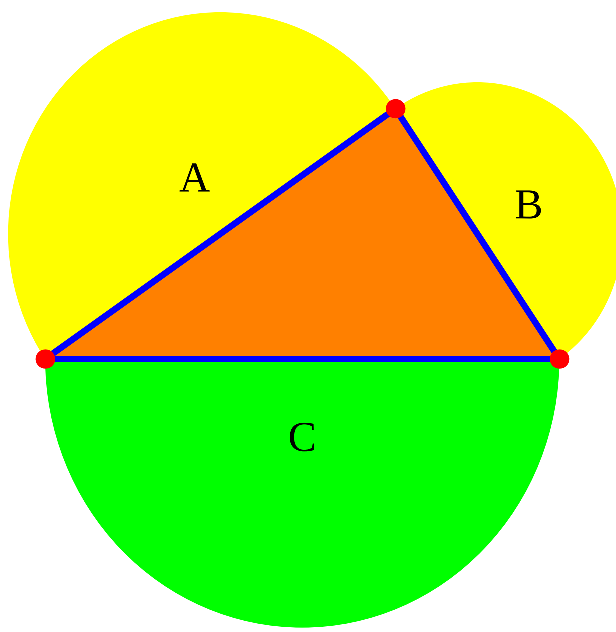
2. Hyppocrates theorem

In this worksheet, we prove the quadrature of the Lune, a result of **Hippocrates of Chios** (470 BC - 400 BC). It is the first rigorous quadrature of a curvilinear area.

The sum $L + R$ of the area L of the left moon and the area R of the right moon is equal to the area T of the triangle.

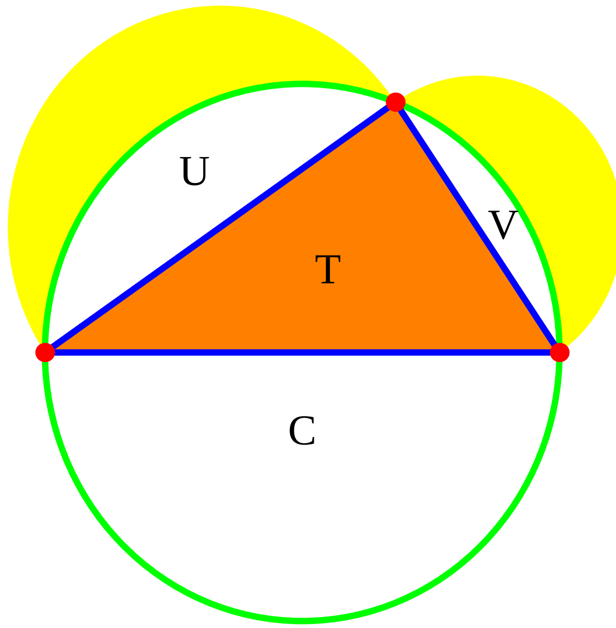


a) Relate first the area of each half circle with the corresponding triangle side length. Let now A, B, C the areas of the half circles build over the sides of the triangle. Show that $A + B = C$. The picture shows the half disc below the hypothenuse.



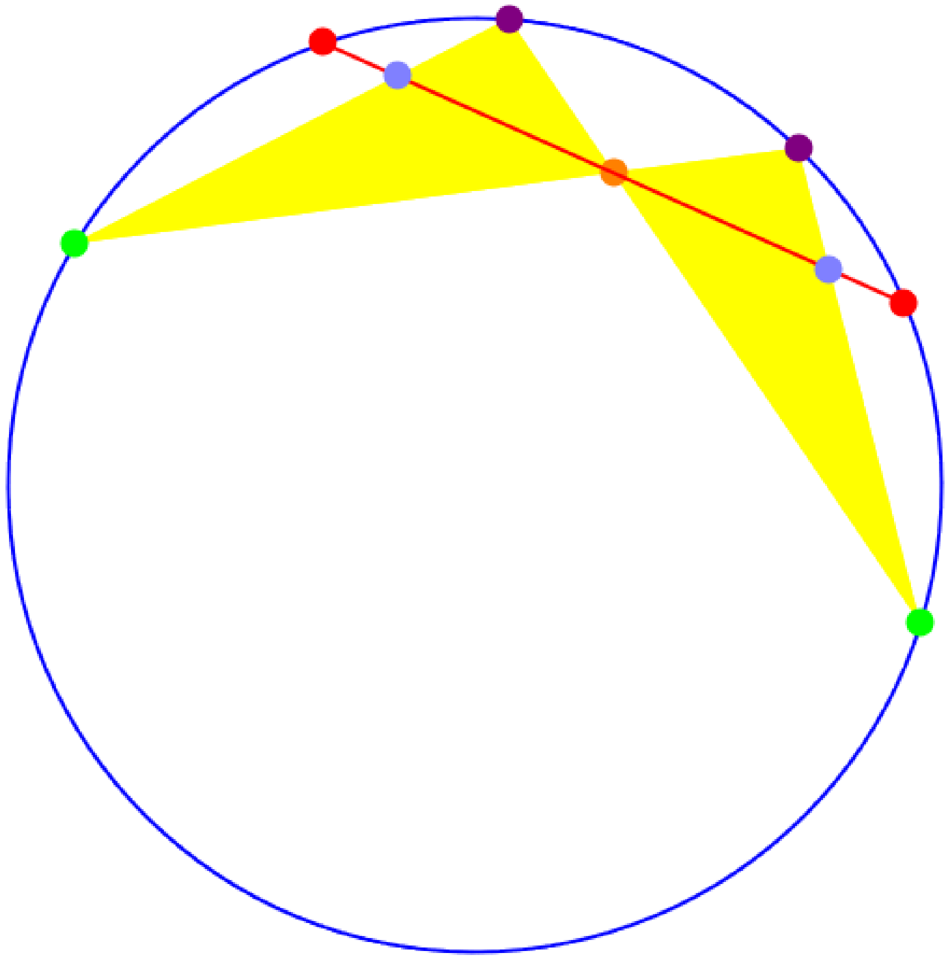
b) Let U be the area of the intersection of A with the upper half circle C. and let V be the area of the intersection of B with C. Let T be the area of the triangle. Why is $U + V + T = C$?

c) The number $A - U$ is the area of a region. Which one. Similarly, what is $B - V$?
Can you finish the proof of the theorem $L + R = T$?



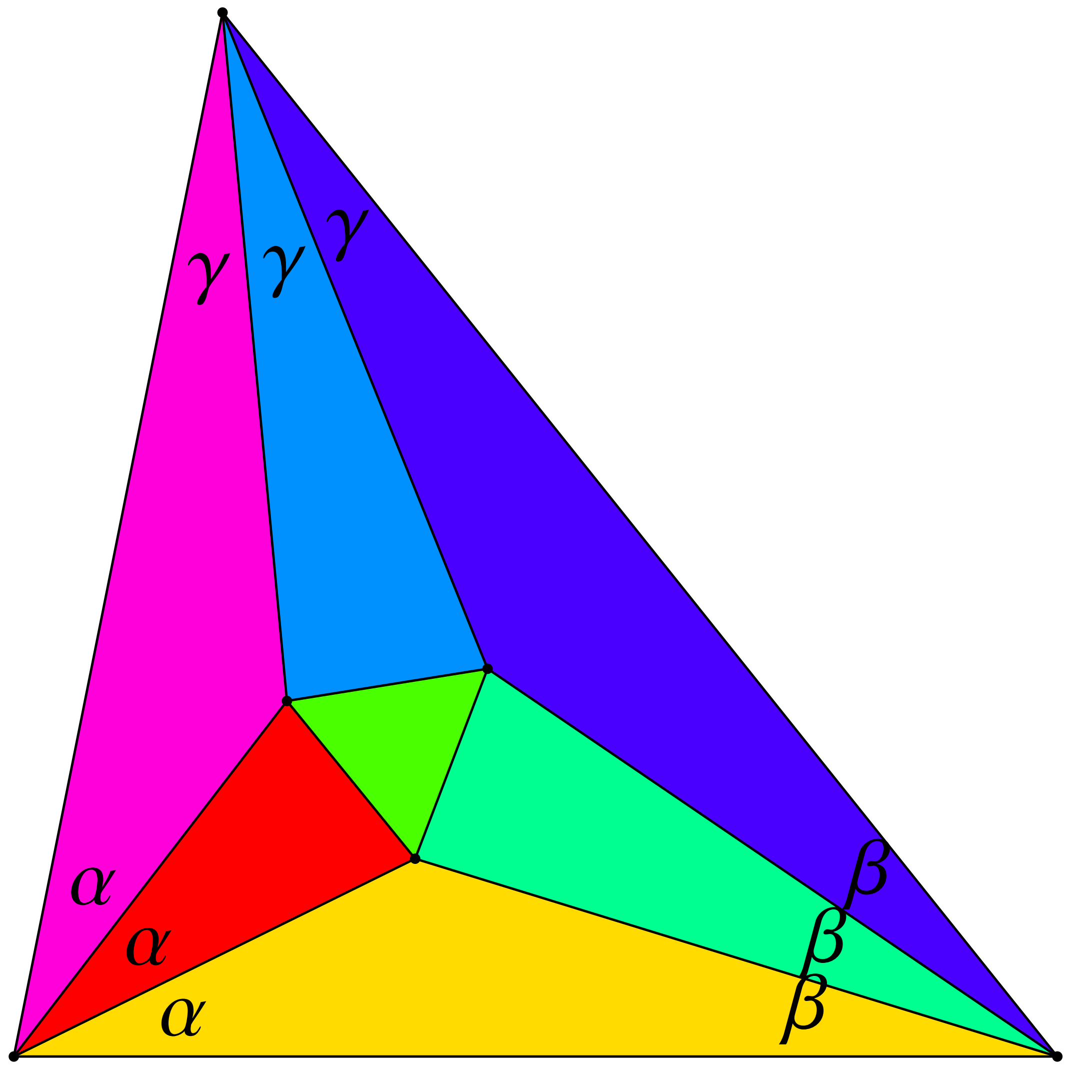
The butterfly theorem

We want to see why the butterfly wings in the butterfly theorem are similar triangles:



Morley's miracle

In order to understand the proof of Morley's miracle we can decompose the triangle with 7 triangles.



Fasskreis theorem

Given a circle of radius 1 and a point P inside the circle. For any line through P which intersects the circle at points A, B we have $1 - |PO|^2 = |PA||PB|$.

Proof with Pythagoras. By scaling translation and rotation we can assume the circle is at the origin and that the line through the point $P = (a, b)$ is vertical. The intersection points are then $(a, \pm\sqrt{1 - a^2})$. Now

$$|PA||PB| = (\sqrt{1 - a^2} - b)(\sqrt{1 - a^2} + b) = 1 - a^2 - b^2 = 1 - |PO|^2 .$$