Lecture 3: Geometry

Geometry is the science of **shape, size and symmetry**. While arithmetic deals with numerical structures, geometry handles metric structures. Geometry is one of the oldest mathematical disciplines. Early geometry has relations with arithmetic: the multiplication of two numbers $n \times m$ as an area of a **shape** that is invariant under rotational **symmetry**. Identities like the **Pythagorean triples** $3^2 + 4^2 = 5^2$ were interpreted and drawn geometrically. The **right angle** is the most "symmetric" angle apart from 0. Symmetry manifests itself in quantities which are **invariant**. Invariants are one the most central aspects of geometry. Felix Klein's **Erlanger program** uses symmetry to classify geometries depending on how large the symmetries of the shapes are. In this lecture, we look at a few results which can all be stated in terms of invariants. In the presentation as well as the worksheet part of this lecture, we will work us through smaller miracles like **special points in triangles** as well as a couple of gems: **Pythagoras**, **Thales**,**Hippocrates**, **Feuerbach**, **Pappus**, **Morley**, **Butterfly** which illustrate the importance of symmetry.

Much of geometry is based on our ability to measure **length**, the **distance** between two points. Having a distance $d(A, B)$ between any two points A, B, we can look at the next more complicated object, which is a set A, B, C of 3 points, a **triangle**. Given an arbitrary triangle ABC, are there relations between the 3 possible distances $a = d(B, C), b = d(A, C), c = d(A, B)$? If we fix the scale by $c = 1$, then $a + b \geq 1, a + 1 \geq b, b + 1 \geq a$. For any pair of (a, b) in this region, there is a triangle. After an identification, we get an abstract space, which represent all triangles uniquely up to similarity. Mathematicians call this an example of a **moduli space**.

A **sphere** $S_r(x)$ is the set of points which have distance r from a given point x. In the plane, the sphere is called a **circle**. A natural problem is to find the circumference $L = 2\pi$ of a unit circle, or the area $A = \pi$ of a unit disc, the area $F = 4\pi$ of a unit sphere and the volume $V = 4 = \pi/3$ of a unit sphere. Measuring the length of segments on the circle leads to new concepts like **angle** or **curvature**. Because the circumference of the unit circle in the plane is $L = 2\pi$, angle questions are tied to the number π, which Archimedes already approximated by fractions.

Also **volumes** were among the first quantities, Mathematicians wanted to measure and compute. A problem on **Moscow papyrus** dating back to 1850 BC explains the general formula $h(a^2+ab+b^2)/3$ for a truncated pyramid with base length a, roof length b and height h. Archimedes achieved to compute the **volume of the sphere**: place a cone inside a cylinder. The complement of the cone inside the cylinder has on each height h the area $\pi - \pi h^2$. The half sphere cut at height h is a disc of radius $(1 - h^2)$ which has area $\pi(1 - h^2)$ too. Since the slices at each height have the same area, the volume must be the same. The complement of the cone inside the cylinder has volume $\pi - \pi/3 = 2\pi/3$, half the volume of the sphere.

The first geometric playground was **planimetry**, the geometry in the flat two dimensional space. Highlights are **Pythagoras theorem**, **Thales theorem**, **Hippocrates theorem**, and **Pappus theorem**. Discoveries in planimetry have been made later on: an example is the Feuerbach 9 point theorem from the 19th century. Ancient Greek Mathematics is closely related to history. It starts with **Thales** goes over Euclid's era at 500 BC and ends with the threefold destruction of Alexandria 47 BC by the Romans, 392 by the Christians and 640 by the Muslims. Geometry was also a place, where the **axiomatic method** was brought to mathematics: theorems are proved from a few statements which are called axioms like the 5 axioms of Euclid:

1. Any two distinct points A, B determines a line through A and B.
2. A line segment $[A, B]$ can be extended to a straight line containing the segment.
3. A line segment $[A, B]$ determines a circle containing B and center A.
4. All right angles are congruent.
5. If lines L, M intersect with a third so that inner angles add up to $< \pi$, then L, M intersect.

Euclid wondered whether the fifth postulate can be derived from the first four and called theorems derived from the first four the "absolute geometry". Only much later, with **Karl-Friedrich Gauss** and **Janos Bolyai** and **Nicolai Lobachevsky** in the 19'th century in **hyperbolic space** the 5'th axiom does not hold. Indeed, geometry can be generalized to non-flat, or even much more abstract situations. Basic examples are geometry on a sphere leading to **spherical geometry** or geometry on the Poincare disc, a **hyperbolic space**. Both of these geometries are non-Euclidean. **Riemannian geometry**, which is essential for **general relativity theory** generalizes both concepts to a great extent. An example is the geometry on an arbitrary surface. Curvatures of such spaces can be computed by measuring length alone, which is how long light needs to go from one point to the next.

An important moment in mathematics was the **merge of geometry with algebra**: this giant step is often attributed to **René Descartes**. Together with algebra, the subject leads to algebraic geometry which can be tackled with computers: here are some examples of geometries which are determined from the amount of symmetry which is allowed:

Euclidean geometry	Properties invariant under a group of rotations and translations
Affine geometry	Properties invariant under a group of affine transformations
Projective geometry	Properties invariant under a group of projective transformations
Spherical geometry	Properties invariant under a group of rotations
Conformal geometry	Properties invariant under angle preserving transformations
Hyperbolic geometry	Properties invariant under a group of Möbius transformations

Here are four pictures about the 4 special points in a triangle and with which we will begin the lecture. We will see why in each of these cases, the 3 lines intersect in a common point. It is a manifestation of a **symmetry** present on the space of all triangles. **size** of the distance of intersection points is constant 0 if we move on the space of all triangular **shapes**. It's Geometry!