Lecture 2: Arithmetic

The oldest mathematical discipline is **arithmetic**. It is the theory of the construction and manipulation of numbers. The earliest steps were done by **Babylonian**, **Egyptian**, **Chinese**, **Indian** and **Greek** thinkers. Building up the number system starts with the **natural numbers** $1, 2, 3, 4...$ which can be added and multiplied. Addition is natural: join 3 sticks to 5 sticks to get 8 sticks. Multiplication $*$ is more subtle: $3 * 4$ means to take 3 copies of 4 and get $4 + 4 + 4 = 12$ while $4 * 3$ means to take 4 copies of 3 to get $3 + 3 + 3 + 3 = 12$. The first factor counts the number of operations while the second factor counts the objects. To motivate $3 * 4 = 4 * 3$, spacial insight motivates to arrange the 12 objects in a rectangle. This commutativity axiom will be carried over to larger number systems. Realizing an addition and multiplicative structure on the natural numbers requires to define 0 and 1. It leads naturally to more general numbers. There are two major motivations to **to build new numbers**: we want to

1. **invert operations** and still get results. 2. **solve equations**.

To find an additive inverse of 3 means solving $x + 3 = 0$. The answer is a negative number. To solve $x * 3 = 1$, we get to a rational number $x = 1/3$. To solve $x^2 = 2$ one need to escape to real numbers. To solve $x^2 = -2$ requires complex numbers.

Numbers	Operation to complete	Examples of equations to solve
Natural numbers	addition and multiplication	$5 + x = 9$
Positive fractions	addition and division	$5x = 8$
Integers	subtraction	$5 + x = 3$
Rational numbers	division	$3x = 5$
Algebraic numbers	taking positive roots	$x^2 = 2$, $2x + x^2 - x^3 = 2$
Real numbers	taking limits	$x = 1 - 1/3 + 1/5 - +...$,$\cos(x) = x$
Complex numbers	take any roots	$x^2 = -2$
Surreal numbers	transfinite limits	$x^2 = \omega$, $1/x = \omega$
Surreal complex	any operation	$x^2 + 1 = -\omega$

The development and history of arithmetic can be summarized as follows: humans started with natural numbers, dealt with positive fractions, reluctantly introduced negative numbers and zero to get the integers, struggled to "realize" real numbers, were scared to introduce complex numbers, hardly accepted surreal numbers and most do not even know about surreal complex numbers. Ironically, as simple but impossibly difficult questions in number theory show, the modern point of view is the opposite to Kronecker's **"God made the integers; all else is the work of man"**:

The **surreal complex** numbers are the most **natural** numbers;
The **natural** numbers are the most **complex, surreal** numbers.

Natural numbers. Counting can be realized by sticks, bones, quipu knots, pebbles or wampum knots. The **tally stick** concept is still used when playing card games: where bundles of fives are formed, maybe by crossing 4 "sticks" with a fifth. There is a "log counting" method in which graphs are used and vertices and edges count. An old stone age tally stick, the **wolf radius bone** contains 55 notches, with 5 groups of 5. It is probably more than 30'000 years old. The most famous paleolithic tally stick is the **Ishango bone**, the fibula of a baboon. It could be 20'000

- 30'000 years old. Earlier counting could have been done by assembling **pebbles**, tying **knots** in a string, making **scratches** in dirt or bark but no such traces have survived the thousands of years. The **Roman system** improved the tally stick concept by introducing new symbols for larger numbers like $V = 5, X = 10, L = 40, C = 100, D = 500, M = 1000$. in order to avoid bundling too many single sticks. The system is unfit for computations as simple calculations $VIII + VII = XV$ show. **Clay tablets**, some as early as 2000 BC and others from 600 - 300 BC are known. They feature **Akkadian arithmetic** using the base 60. The hexadecimal system with base 60 is convenient because of many factors. It survived: we use 60 minutes per hour. **The Egyptians** used the base 10. The most important source on Egyptian mathematics is the **Rhind Papyrus** of 1650 BC. Hieratic numerals were used to write on papyrus from 2500 BC on. **Egyptian numerals** are hieroglyphics. Found in carvings on tombs and monuments they are 5000 years old. The modern way to write numbers like 2017 is the **Hindu-Arab system** which diffused to the West only during the late Middle ages. It replaced the more primitive **Roman system**. Greek arithmetic used a number system with no place values: 9 Greek letters for $1, 2, \dots 9$, nine for $10, 20, \dots, 90$ and nine for $100, 200, \dots, 900$.

Integers. **Indian Mathematics** morphed the place-value system into a modern method of writing numbers. Hindu astronomers used words to represent digits, but the numbers would be written in the opposite order. Independently, also the Mayans developed the concept of 0 in a number system using base 20. Sometimes after 500, the Hindus changed to a digital notation which included the symbol 0. Negative numbers were introduced around 100 BC in the **Chinese** text "Nine Chapters on the Mathematica art". Also the **Bakshali manuscript**, written around 300 AD subtracts numbers carried out additions with negative numbers, where + was used to indicate a negative sign. In Europe, negative numbers were avoided until the 15'th century.

Fractions: **Babylonians** could handle fractions. The **Egyptians** also used fractions, but wrote every fraction a as a sum of fractions with unit numerator and distinct denominators, like $4/5 = 1/2 + 1/4 + 1/20$ or $5/6 = 1/2 + 1/3$. Maybe because of such cumbersome computation techniques, Egyptian mathematics failed to progress beyond a primitive stage. The modern decimal fractions used nowadays for numerical calculations were adopted only in 1595 in Europe.

Real numbers: As noted by the Greeks already, the diagonal of the square is not a fraction. It first produced a crisis until it became clear that "most" numbers are not rational. **Georg Cantor** saw first that the cardinality of all real numbers is much larger than the cardinality of the integers: while one can count all rational numbers but not enumerate all real numbers. One consequence is that most real numbers are transcendental: they do not occur as solutions of polynomial equations with integer coefficients. The number π is an example. The concept of real numbers is related to the **concept of limit**. Sums like $1 + 1/4 + 1/9 + 1/16 + 1/25 + ...$ are not rational.
Complex numbers: Some polynomials have no real root. To solve $x^2 = -1$ for example, we need new numbers. One idea is to use pairs of numbers (a, b) where $(a, 0) = a$ are the usual numbers and extend addition and multiplication $(a, b)+(c, d) = (a+c, b+d)$ and $(a, b)\cdot(c, d) = (ac-bd, ad+bc)$. With this multiplication, the number $(0, 1)$ has the property that $(0, 1) \cdot (0, 1) = (-1, 0) = -1$. It is more convenient to write $a + ib$ where $i = (0, 1)$ satisfies $i^2 = -1$. One can now use the common rules of addition and multiplication.
Surreal numbers: Similarly as real numbers fill in the gaps between the integers, the surreal numbers fill in the gaps between Cantors ordinal numbers. They are written as $(a, b, c, ...|d, e, f, ...)$ meaning that the "simplest" number is larger than $a, b, c...$ and smaller than $d, e, f, ...$ We have $(|) = 0, (0|) = 1, (1|) = 2$ and $(0|1) = 1/2$ or $(|0) = -1$. Surreals contain already transfinite numbers like $(0, 1, 2, 3...|)$ or infinitesimal numbers like $(0|1/2, 1/3, 1/4, 1/5, ...)$. They were introduced in the 1970'ies by John Conway. The late appearance confirms the pedagogical principle: **late human discovery manifests in increased difficulty to teach it**.